

# UNIPO TENT ELEMENTS IN SMALL CHARACTERISTIC, III

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## 0. INTRODUCTION

Let  $\mathbf{k}$  be an algebraically closed field of characteristic exponent  $p \geq 1$ . Let  $G$  be a connected reductive algebraic group over  $\mathbf{k}$  and let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Note that  $G$  acts on  $G$  and on  $\mathfrak{g}$  by the adjoint action. Let  $\mathcal{U}_G$  be the variety of unipotent elements of  $G$ . Let  $\mathcal{N}_{\mathfrak{g}}$  be the variety of nilpotent elements of  $\mathfrak{g}$ . In [L2], we have proposed a definition of a partition of  $\mathcal{U}_G$  into smooth locally closed  $G$ -stable pieces which are indexed by the unipotent classes in the group over  $\mathbf{C}$  of the same type as  $G$  and which in many ways seem to depend very smoothly on  $p$ ; moreover we studied in detail the pieces of  $\mathcal{U}_G$  for types  $A$  and  $C$ . In [L3] we have studied in detail the pieces of  $\mathcal{U}_G$  for types  $B$  and  $D$ ; however the definition in [L3] was not based on the proposal of [L2] (which involved the partial order of unipotent classes).

In this paper we propose another general definition of the pieces of  $\mathcal{U}_G$  which is not based on the partial order of unipotent classes and we show that this new definition unifies the definitions in [L2], [L3] in the sense that for types  $A, C$  it can be identified with the definition in [L2] while for types  $B, D$  it can be identified with the definition in [L3]. The idea of the new definition is as follows. One needs to consider a grading  $\mathfrak{g} = \bigoplus_n \mathfrak{g}_n$  analogous to the one associated to a nilpotent element over  $\mathbf{C}$  by the Morozov-Jacobson theorem. Let  $G_0$  be the closed connected subgroup of  $G$  corresponding to  $\mathfrak{g}_0$ . Now  $G_0$  acts naturally on  $\mathfrak{g}_2$ . The main ingredient in the definition of a piece is the definition of a suitable open  $G_0$ -invariant subset  $\mathfrak{g}_2^!$  of  $\mathfrak{g}_2$ . When  $p = 1$  or  $p \gg 0$ , the subset  $\mathfrak{g}_2^!$  is by definition the unique open  $G_0$ -orbit in  $\mathfrak{g}_2$ . But this definition is not correct for general  $p$ . Now note that when  $p = 1$  the centralizer in  $G$  of any element of the open  $G_0$ -orbit in  $\mathfrak{g}_2$  is contained in  $G_{\geq 0}$ , the parabolic subgroup of  $G$  whose Lie algebra is  $\sum_{i \geq 0} \mathfrak{g}_i$  (a result of Kostant [K]). Based on this, we propose to define (for general  $p$ ) the set  $\mathfrak{g}_2^!$  as the set of all  $x \in \mathfrak{g}_2$  such that the centralizer of  $x$  in  $G$  is contained in  $G_{\geq 0}$ . It turns out that (at least in types  $A, B, C, D$ ) this condition gives exactly the unique open  $G_0$ -orbit in  $\mathfrak{g}_2$  when  $p = 1$  or  $p \gg 0$ , while in general it defines

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a subset of  $\mathfrak{g}_2$  which is a union of possibly several  $G_0$ -orbits (their number is a power of 2) but which is exactly what is needed to define the pieces of  $\mathcal{U}_G$ .

This paper is organized as follows. In Section 1 we give the definition of the sets  $\mathfrak{g}_2^!$  and show that in types  $A, B, C, D$  these sets can be identified with certain explicit subsets of  $\mathfrak{g}_2$  considered in [L2], [L3]. In Section 2 we use the sets  $\mathfrak{g}_2^!$  to define the pieces of  $\mathcal{U}_G$ . In the appendix (written by the author and T. Xue) we define (at least in types  $A, B, C, D$ ) a partition of  $\mathcal{N}_{\mathfrak{g}}$  into smooth locally closed  $G$ -stable pieces which are indexed by the unipotent classes in the group over  $\mathbf{C}$  of the same type as  $G$  (using again the subsets  $\mathfrak{g}_2^!$  defined in Section 1).

*Notation.* The cardinal of a finite set  $X$  is denoted by  $|X|$ .

*Errata to [L1].*

On p.206, the last sentence in 6.8, "For classical types...classes", should be removed.

*Errata to [L2].*

On p.452 line -5 replace  $\Pi \cup \Theta$  by  $\Theta$ .

On p.452 line -4 replace  $\Pi \cup \tilde{\Theta}$  by  $\tilde{\Theta}$ .

*Errata to [L3].*

On p.774 line 6: replace "an (injective)" by "a".

On p.778, line 13: replace last  $T^a$  by  $T^a x$ .

On p.779, line 10: after "(b) and (c)" insert: "and  $f_a = f_{-a}$  for all  $a$ ".

On p.797, remove last line of 4.2 and last line of 4.3.

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1. The sets  $\mathfrak{g}_2^{\delta!}$ .

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Appendix (by G. Lusztig and T. Xue): The pieces in the nilpotent variety of  $\mathfrak{g}$ .

### 1. THE SETS $\mathfrak{g}_2^{\delta!}$

**1.1.** Let  $\mathbf{T}, \mathbf{W}$  be "the maximal torus" and "the Weyl group" of  $G$  and let  $Y_G = \text{Hom}(\mathbf{k}^*, \mathbf{T})$ . Note that  $\mathbf{W}$  acts naturally on  $\mathbf{T}$  and on  $Y_G$ .

Let  $G'$  be a connected reductive algebraic group over  $\mathbf{C}$  of the same type as  $G$ . In particular we have canonically  $Y_G = Y_{G'}$  compatibly with the  $\mathbf{W}$ -actions. Note that  $G$  acts by conjugation on  $\text{Hom}(\mathbf{k}^*, G)$ ; similarly,  $G'$  acts by conjugation on  $\text{Hom}(\mathbf{C}^*, G')$ . We have canonically  $G \backslash \text{Hom}(\mathbf{k}^*, G) = \mathbf{W} \backslash Y_G$ ,  $G' \backslash \text{Hom}(\mathbf{C}^*, G') = \mathbf{W} \backslash Y_{G'}$ . Let  $\mathfrak{D}_{G'}$  be the set of all  $f \in \text{Hom}(\mathbf{C}^*, G')$  such that for some homomorphism of algebraic groups  $h : SL_2(\mathbf{C}) \rightarrow G'$  we have  $h \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = f(a)$  for all  $a \in \mathbf{C}^*$ . Let  $\mathfrak{D}_G$  be the set of all  $\delta \in \text{Hom}(\mathbf{k}^*, G)$  such that the image of  $\delta$  in  $G \backslash \text{Hom}(\mathbf{k}^*, G) = \mathbf{W} \backslash Y_G = \mathbf{W} \backslash Y_{G'} = G' \backslash \text{Hom}(\mathbf{C}^*, G')$  can be represented by an element in  $\mathfrak{D}_{G'} \subset \text{Hom}(\mathbf{C}^*, G')$ . (See [L2, 1.1].) Note that  $\mathfrak{D}_G$  is a union of  $G$ -orbits. The  $G'$ -orbits in  $\mathfrak{D}_{G'}$  were classified by Dynkin. They form a finite set in natural bijection with the set of  $G$ -orbits in  $\mathfrak{D}_G$ .

Let  $G^{der}$  be the derived group of  $G$ . The simply connected covering  $\tilde{G}^{der} \rightarrow G^{der}$  induces a bijection  $\mathfrak{D}_{\tilde{G}^{der}} \rightarrow \mathfrak{D}_G$ . This follows from the analogous assertion for  $G'$  which is immediate.

**1.2.** Let  $\delta \in \text{Hom}(\mathbf{k}^*, G)$ . For any  $i \in \mathbf{Z}$  we set

$$\mathfrak{g}_i^\delta = \{x \in \mathfrak{g}; \text{Ad}(\delta(a))x = a^i x \quad \forall a \in \mathbf{k}^*\}.$$

We have a direct sum decomposition  $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}_i^\delta$ . For  $i \in \mathbf{N}$  we set  $\mathfrak{g}_{\geq i}^\delta = \bigoplus_{j \in \mathbf{Z}; j \geq i} \mathfrak{g}_j^\delta$ ; note that  $\mathfrak{g}_{\geq i}^\delta$  is the Lie algebra of a well defined closed connected subgroup  $G_{\geq i}^\delta$  of  $G$ . We have  $\dots \subset G_{\geq 2}^\delta \subset G_{\geq 1}^\delta \subset G_{\geq 0}^\delta$  and  $G_{\geq 0}^\delta$  is a parabolic subgroup of  $G$  with unipotent radical  $G_{\geq 1}^\delta$  and with Levi subgroup  $G_0^\delta$  (with Lie algebra  $\mathfrak{g}_0^\delta$ ); moreover, for any  $i$ ,  $G_{\geq i}^\delta$  is a normal subgroup of  $G_{\geq 0}^\delta$ .

We set  $\mathfrak{g}_{<0}^\delta = \bigoplus_{j \in \mathbf{Z}; j < 0} \mathfrak{g}_j^\delta$ ; note that  $\mathfrak{g}_{<0}^\delta$  is the Lie algebra of a well defined closed connected subgroup  $G_{<0}^\delta$  of  $G$ . We have  $G_{<0}^\delta \cap G_{\geq 0}^\delta = \{1\}$ .

For any  $x \in \mathfrak{g}$  let  $G_x = \{g \in G; \text{Ad}(g)x = x\}$ . Let

$$\mathfrak{g}_2^{\delta!} = \{x \in \mathfrak{g}_2^\delta; G_x \subset G_{\geq 0}^\delta\}.$$

For  $h \in G_0^\delta$ ,  $x \in \mathfrak{g}$  we have  $G_{\text{Ad}(h)x} = hG_x h^{-1}$ ; hence  $\text{Ad}(h)\mathfrak{g}_2^{\delta!} = \mathfrak{g}_2^{\delta!}$ . Thus,  $\mathfrak{g}_2^{\delta!}$  is a union of orbits for the  $\text{Ad}$ -action of  $G_0^\delta$  on  $\mathfrak{g}_2^\delta$ .

In 1.3, 1.4, 1.5 we will describe explicitly the set  $\mathfrak{g}_2^{\delta!}$  in a number of cases.

**1.3.** In this subsection we fix a  $\mathbf{k}$ -vector space  $V$  of finite dimension and we assume that  $G = GL(V)$ . Then  $\mathfrak{g} = \text{End}(V)$ . A  $\mathbf{Z}$ -grading  $V = \bigoplus_{i \in \mathbf{Z}} V_i$  of  $V$  is said to be *good* if  $\dim V_i = \dim V_{-i} \geq \dim V_{-i-2}$  for all  $i \geq 0$ . If a good  $\mathbf{Z}$ -grading  $(V_i)$  is given and  $a \in \mathbf{Z}$ , we set

$$\begin{aligned} V_{\geq a} &= \sum_{j \geq a} V_j, \\ \text{End}(V)_a &= \{A \in \mathfrak{g}; A(V_r) \subset V_{r+a} \quad \forall r \in \mathbf{Z}\}, \\ \text{End}(V)_{\geq a} &= \{A \in \mathfrak{g}; A(V_{\geq r}) \subset V_{\geq r+a} \quad \forall r \in \mathbf{Z}\}. \end{aligned}$$

To give an element  $\delta \in \mathfrak{D}_G$  is the same as to give a good grading  $V = \bigoplus_{i \in \mathbf{Z}} V_i$  of  $V$  ( $\delta$  is given in terms of the  $\mathbf{Z}$ -grading by  $\delta(a)|_{V_r} = a^r$  for all  $a \in \mathbf{k}^*, r \in \mathbf{Z}$ .)

In the remainder of this subsection we fix  $\delta \in \mathfrak{D}_G$  and let  $(V_i)$  be the corresponding good grading of  $V$ . Then  $\mathfrak{g}_i^\delta = \text{End}(V)_i$ . Let  $\text{End}(V)_2^0$  be the set of all  $A \in \text{End}(V)_2$  such that  $A^n : V_{-n} \rightarrow V_n$  is an isomorphism for any  $n \geq 0$ . It is easy to see that  $\text{End}(V)_2^0 \neq \emptyset$ . The following result will be proved in 1.6.

$$(a) \quad \mathfrak{g}_2^{\delta!} = \text{End}(V)_2^0.$$

**1.4.** In this subsection we fix a  $\mathbf{k}$ -vector space  $V$  of finite even dimension with a fixed nondegenerate symplectic form  $(, ) : V \times V \rightarrow \mathbf{k}$  and we assume that  $G = Sp(V) = \{T \in GL(V); T \text{ preserves } (, )\}$ . Let  $\mathfrak{s}(V) = \{T \in \text{End}(V); (Tv, v') + (v, Tv') = 0 \quad \forall v, v' \in V\}$ . Then  $\mathfrak{g} = \mathfrak{s}(V)$ .

A  $\mathbf{Z}$ -grading  $V = \bigoplus_{i \in \mathbf{Z}} V_i$  of  $V$  is said to be *s-good* if it is good (see 1.3),  $\dim V_i$  is even for any even  $i$  and  $(V_i, V_j) = 0$  whenever  $i + j \neq 0$ .

To give an element  $\delta \in \mathfrak{D}_G$  is the same as to give an *s-good* grading  $V = \bigoplus_{i \in \mathbf{Z}} V_i$  of  $V$  ( $\delta$  is given in terms of the  $\mathbf{Z}$ -grading by  $\delta(a)|_{V_r} = a^r$  for all  $a \in \mathbf{k}^*, r \in \mathbf{Z}$ .)

In the remainder of this subsection we fix  $\delta \in \mathfrak{D}_G$  and let  $(V_i)$  be the corresponding  $s$ -good grading of  $V$ . Let  $\mathfrak{s}(V)_i = \mathfrak{s}(V) \cap \text{End}(V)_i$ . Then  $\mathfrak{g}_i^\delta = \mathfrak{s}(V)_i$  for any  $i$ . Let  $\mathfrak{s}(V)_2^0 = \mathfrak{s}(V)_2 \cap \text{End}(V)_2^0$  (notation of 1.3). The following result will be proved in 1.7.

$$(a) \mathfrak{g}_2^{\delta!} = \mathfrak{s}(V)_2^0.$$

**1.5.** In this subsection we fix a  $\mathbf{k}$ -vector space  $V$  of finite dimension with a fixed nondegenerate quadratic form  $Q : V \rightarrow \mathbf{k}$  with associate symmetric bilinear form  $(, ) : V \times V \rightarrow \mathbf{k}$ . (Recall that  $(v, v') = Q(v + v') - Q(v) - Q(v')$  for  $v, v' \in V$  and that the nondegeneracy of  $Q$  means that, if  $\mathcal{R}$  is the radical of  $(, )$ , then  $\mathcal{R} = 0$  if  $p \neq 2$  and  $Q : \mathcal{R} \rightarrow \mathbf{k}$  is injective if  $p = 2$ .) We assume that  $G = SO(V)$ , the identity component of  $O(V) = \{T \in GL(V); Q(T(v)) = Q(v) \ \forall v \in V\}$ . Let  $\mathfrak{o}(V) = \{T \in \text{End}(V); (Tv, v) = 0 \ \forall v \in V, T|_{\mathcal{R}} = 0\}$ . Then  $\mathfrak{g} = \mathfrak{o}(V)$ .

A  $\mathbf{Z}$ -grading  $V = \bigoplus_{i \in \mathbf{Z}} V_i$  of  $V$  is said to be *o-good* if it is good (see 1.3),  $\dim V_i$  is even for any odd  $i$ ,  $(V_i, V_j) = 0$  whenever  $i + j \neq 0$  and  $Q|_{V_i} = 0$  whenever  $i \neq 0$ .

To give an element  $\delta \in \mathfrak{D}_G$  is the same as to give an *o-good* grading  $V = \bigoplus_{i \in \mathbf{Z}} V_i$  of  $V$  ( $\delta$  is given in terms of the  $\mathbf{Z}$ -grading by  $\delta(a)|_{V_r} = a^r$  for all  $a \in \mathbf{k}^*, r \in \mathbf{Z}$ .)

In the remainder of this subsection we fix  $\delta \in \mathfrak{D}_G$  and let  $(V_i)$  be the corresponding *o-good* grading of  $V$ . Let  $\mathfrak{o}(V)_i = \mathfrak{o}(V) \cap \text{End}(V)_i$ . Then  $\mathfrak{g}_i^\delta = \mathfrak{o}(V)_i$  for any  $i$ . Let  $\mathfrak{o}(V)_2^0$  be the set of all  $A \in \mathfrak{o}(V)_2$  such that

(i) for any odd  $n \geq 1$ ,  $A^n : V_{-n} \rightarrow V_n$  is an isomorphism;

(ii) for any even  $n \geq 0$ ,  $A^{n/2} : V_{-n} \rightarrow V_0$  is injective and the restriction of  $Q$  to  $A^{n/2}(V_{-n})$  is nondegenerate.

Note that the equality  $\mathfrak{o}(V)_2^0 = \mathfrak{o}(V)_2 \cap \text{End}(V)_2^0$  (notation of 1.3) holds when  $p \neq 2$ , but not necessarily when  $p = 2$ . The following result will be proved in 1.8.

$$(a) \mathfrak{g}_2^{\delta!} = \mathfrak{o}(V)_2^0.$$

**1.6.** We prove 1.3(a). Generally,  $x_k$  will denote an element of  $V_k$ . Let  $A \in \text{End}(V)_2 - \text{End}(V)_2^0$ . The transpose  $A^* : V^* \rightarrow V^*$  of  $A$  carries  $V_i^*$  to  $V_{i-2}^*$ .

Assume first that  $A : V_{-i} \rightarrow V_{-i+2}$  is not injective for some  $i \geq 2$ . Then  $A^* : V_{-i+2}^* \rightarrow V_{-i}^*$  is not surjective. Since  $\dim V_{-i+2} \geq \dim V_{-i}$  it follows that  $A^* : V_{-i+2}^* \rightarrow V_{-i}^*$  is not injective. We can find  $e_{-i} \in V_{-i} - \{0\}$  such that  $Ae_{-i} = 0$  and  $\xi_{-i+2} \in V_{-i+2}^* - \{0\}$  such that  $A^*\xi_{-i+2} = 0$ . Define  $B \in \text{End}(V)$  by

$$B(\sum_k x_k) = \sum_{k \neq -i} x_k + (x_{-i} + \xi_{-i+2}(x_{-i+2})e_{-i}).$$

We have  $B \in G_{<0}^\delta - \{1\}$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned} BA(\sum_k x_k) - AB(\sum_k x_k) &= \sum_{k \neq -i} Ax_{k-2} + (Ax_{-i-2} + \xi_{-i+2}(Ax_{-i})e_{-i}) \\ &\quad - \sum_{k \neq -i} x_k - (Ax_{-i} + \xi_{-i+2}(x_{-i+2})Ae_{-i}) \\ &= (A^*\xi_{-i+2})(x_{-i})e_{-i} - \xi_{-i+2}(x_{-i+2})Ae_{-i} = 0. \end{aligned}$$

Thus  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

Assume next that  $A : V_i \rightarrow V_{i+2}$  is not surjective for some  $i \geq 0$ . Since  $\dim V_i \geq \dim V_{i+2}$ ,  $A : V_i \rightarrow V_{i+2}$  is not injective. Moreover  $A^* : V_{i+2}^* \rightarrow V_i^*$  is not injective. We can find  $e_i \in V_i - \{0\}$  such that  $Ae_i = 0$  and  $\xi_{i+2} \in V_{i+2}^* - \{0\}$  such that  $A^*\xi_{i+2} = 0$ . Define  $B \in \text{End}(V)$  by

$$B(\sum_k x_k) = \sum_{k \neq i} x_k + (x_i + \xi_{i+2}(x_{i+2})e_i).$$

We have  $B \in G_{<0}^\delta - \{1\}$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned} BA(\sum_k x_k) - AB(\sum_k x_k) &= \sum_{k \neq i} Ax_{k-2} + (Ax_{i-2} + \xi_{i+2}(Ax_i)e_i) \\ &\quad - \sum_{k \neq i} x_k - (Ax_i + \xi_{i+2}(x_{i+2})Ae_i) = (A^*\xi_{i+2})(x_i)e_i - \xi_{i+2}(x_{i+2})Ae_i = 0. \end{aligned}$$

Thus  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We now assume that  $A : V_{-i} \rightarrow V_{-i+2}$  is injective for any  $i \geq 2$  and  $A : V_i \rightarrow V_{i+2}$  is surjective for any  $i \geq 0$ . For some  $n > 0$ ,  $A^n : V_{-n} \rightarrow V_n$  is not an isomorphism hence  $A^{*n} : V_n^* \rightarrow V_{-n}^*$  is not an isomorphism. We can find  $e_{-n} \in V_{-n} - \{0\}$  such that  $A^n e_{-n} = 0$ . We can find  $\xi_n \in V_n^* - \{0\}$  such that  $A^{*n} \xi_n = 0$ . For any  $j \geq 0$  we set  $e_{2j-n} = A^j e_{-n} \in V_{2j-n}$ ,  $\xi_{n-2j} = A^{*j} \xi_n \in V_{n-2j}^*$ . Note that  $e_n = 0, \xi_{-n} = 0$ . Also,  $e_m \neq 0$  if  $m \leq 0$ ,  $m = n \pmod 2$  and  $\xi_m \neq 0$  if  $m \geq 0$ ,  $m = n \pmod 2$ . Define  $B \in \text{End}(V)$  by

$$B(\sum_k x_k) = \sum_{k \notin \{2h-n; h \in [0, n-1]\}} x_k + \sum_{j \in [0, n-1]} (x_{2j-n} + \xi_{2j-n+2}(x_{2j-n+2})e_{2j-n}).$$

We have  $B \in G_{<0}^\delta$ . If  $n$  is even then the term corresponding to  $j = n/2 - 1$  is  $x_{-2} + \xi_0(x_0)e_{-2}$  and  $\sum_k x_k \mapsto \xi_0(x_0)e_{-2}$  is  $\neq 0$  since  $e_{-2} \neq 0, \xi_0 \neq 0$ . If  $n$  is odd then the term corresponding to  $j = (n-1)/2$  is  $x_{-1} + \xi_1(x_1)e_{-1}$  and  $\sum_k x_k \mapsto \xi_1(x_1)e_{-1}$  is  $\neq 0$  since  $e_{-1} \neq 0, \xi_1 \neq 0$ . Thus  $B \neq 1$  so that  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned} BA(\sum_k x_k) - AB(\sum_k x_k) &= \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_{k-2} + \sum_{j \in [0, n-1]} (Ax_{2j-n-2} + \xi_{2j-n+2}(Ax_{2j-n})e_{2j-n}) \\ &\quad - \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_k - \sum_{j \in [0, n-1]} (Ax_{2j-n} + \xi_{2j-n+2}(x_{2j-n+2})Ae_{2j-n}) \\ &= \sum_{j \in [0, n-1]} (A^*\xi_{2j-n+2})(x_{2j-n})e_{2j-n} - \sum_{j \in [0, n-1]} \xi_{2j-n+2}(x_{2j-n+2})Ae_{2j-n} \\ &= \sum_{j \in [0, n-1]} \xi_{2j-n}(x_{2j-n})e_{2j-n} - \sum_{j \in [0, n-1]} \xi_{2j-n+2}(x_{2j-n+2})e_{2j-n+2} \\ &= \sum_{j \in [0, n-1]} \xi_{2j-n}(x_{2j-n})e_{2j-n} - \sum_{j \in [1, n]} \xi_{2j-n}(x_{2j-n})e_{2j-n} \\ &= \xi_{-n}(x_{-n})e_{-n} - \xi_n(x_n)e_n = 0 \end{aligned}$$

since  $x_{-n} = 0, e_n = 0$ . Thus  $BA = AB$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We have shown that  $\text{End}(V)_2 - \text{End}(V)_2^0 \subset \text{End}(V)_2 - \mathfrak{g}_2^{\delta!}$ .

Conversely, let  $A \in \text{End}(V)_2^0$ . We show that  $A \in \mathfrak{g}_2^{\delta!}$ . Let  $B \in G$  be such that  $AB = BA$ . It is enough to show that  $B \in G_{\geq 0}^\delta$ . We argue by induction on  $\dim V$ . If  $V = 0$  the result is clear. Assume now that  $V \neq 0$ . Let  $m$  be the largest integer  $\geq 0$  such that  $V_m \neq 0$ . If  $m = 0$  we have  $G_{\geq 0}^\delta = G$  and the result is clear. Assume now that  $m \geq 1$ . We have  $A^m V = V_m$ ,  $\ker(A^m : V \rightarrow V) = V_{\geq -m+1}$ . Since  $BA = AB$  we have  $B(A^m V) = A^m V$ ,  $B(\ker(A^m : V \rightarrow V)) = \ker(A^m : V \rightarrow V)$ . Hence  $B(V_m) = V_m$  and  $B(V_{\geq -m+1}) = V_{\geq -m+1}$ . Hence  $B$  induces an automorphism  $B' : V' \rightarrow V'$  where  $V' = V_{\geq -m+1}/V_m$ . We have canonically  $V' = V_{-m+1} \oplus V_{-m+2} \oplus \dots \oplus V_{m-1}$  and  $\text{End}(V')_2, \text{End}(V')_2^0$  are defined in terms of this (good) grading. Now  $A$  induces an element  $A' \in \text{End}(V')_2^0$  and we have  $B'A' = A'B'$ . By the induction hypothesis, for any  $i \in [-m+1, m-1]$ , the subspace  $V_i + V_{i+1} + \dots + V_{m-1}$  of  $V'$  is  $B'$ -stable. Hence the subspace  $V_i + V_{i+1} + \dots + V_{m-1} + V_m$  of  $V$  is  $B$ -stable. We see that  $B \in G_{\geq 0}^\delta$ . This completes the proof of 1.3(a).

**1.7.** We prove 1.4(a). Generally,  $x_k$  will denote an element of  $V_k$ .

Let  $A \in \mathfrak{s}(V)_2 - \mathfrak{s}(V)_2^0$ . Assume first that  $A : V_{-i} \rightarrow V_{-i+2}$  is not injective for some  $i > 2$ . Then  $A : V_{i-2} \rightarrow V_i$  is not surjective and since  $\dim V_{i-2} \geq \dim V_i$ , we see that  $A : V_{i-2} \rightarrow V_i$  is not injective. We can find  $e_{-i} \in V_{-i} - \{0\}$  such that  $Ae_{-i} = 0$ . We can find  $e_{i-2} \in V_{i-2} - \{0\}$  such that  $Ae_{i-2} = 0$ . Define  $B \in \text{End}(V)$  by

$$B\left(\sum_k x_k\right) = \sum_{k \neq -i, i-2} x_k + (x_{-i} + (e_{i-2}, x_{-i+2})e_{-i}) + (x_{i-2} + (e_{-i}, x_i)e_{i-2})$$

where  $x_k \in V_k$ . We have

$$\begin{aligned} & (B(\sum_k x_k), B(\sum_k x'_k)) - (\sum_k x_k, \sum_k x'_k) \\ &= (x_{-i} + (e_{i-2}, x_{-i+2})e_{-i}, x'_i) + (x_i, x'_{-i} + (e_{i-2}, x'_{-i+2})e_{-i}) \\ &+ (x_{i-2} + (e_{-i}, x_i)e_{i-2}, x'_{-i+2}) + (x_{-i+2}, x'_{i-2} + (e_{-i}, x'_i)e_{i-2}) \\ &+ \sum_{k \neq -i+2, -i, i-2} (x_{-k}, x'_k) - \sum_k (x_{-k}, x'_k) \\ &= (e_{i-2}, x_{-i+2})(e_{-i}, x'_i) + (x_i, e_{-i})(e_{i-2}, x'_{-i+2}) \\ &+ (e_{-i}, x_i)(e_{i-2}, x'_{-i+2}) + (x_{-i+2}, e_{i-2})(e_{-i}, x'_i) \\ &= 0. \end{aligned}$$

Thus,  $B \in Sp(V)$ . More precisely,  $B \in G_{<0}^\delta - \{1\}$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned}
& BA\left(\sum_k x_k\right) - AB\left(\sum_k x_k\right) \\
&= \sum_{k \neq -i, i-2} Ax_{k-2} + (Ax_{-i-2} + (e_{i-2}, Ax_{-i})e_{-i}) + (Ax_{i-4} + (e_{-i}, Ax_{i-2})e_{i-2}) \\
&- \sum_{k \neq -i, i-2} Ax_k - (Ax_{-i} + (e_{i-2}, x_{-i+2})Ae_{-i}) - (Ax_{i-2} + (e_{-i}, x_i)Ae_{i-2}) \\
&= -(Ae_{i-2}, x_{-i})e_{-i} - (Ae_{-i}, x_{i-2})e_{i-2} - (e_{i-2}, x_{-i+2})Ae_{-i} - (e_{-i}, x_i)Ae_{i-2} = 0. \blacksquare
\end{aligned}$$

Thus,  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

Next we assume that  $A : V_{-2} \rightarrow V_0$  is not injective. We can find  $e_{-2} \in V_{-2} - \{0\}$  such that  $Ae_{-2} = 0$ . Note that  $K' = \ker(A : V_0 \rightarrow V_2) \neq 0$ . (Indeed  $A : V_0 \rightarrow V_2$  is the transpose of  $A : V_{-2} \rightarrow V_0$  hence is not surjective. But  $\dim V_0 \geq \dim V_2$  hence  $K' \neq 0$ .) We can find  $e_0 \in V_0 - \{0\}$  such that  $Ae_0 = 0$ . Define  $B \in \text{End}(V)$  by

$$B\left(\sum_k x_k\right) = \sum_{k \neq -2, 0} x_k + (x_{-2} + (e_0, x_0)e_{-2}) + (x_0 + (e_{-2}, x_2)e_0).$$

We have

$$\begin{aligned}
& (B\left(\sum_k x_k\right), B\left(\sum_k x'_k\right)) - \left(\sum_k x_k, \sum_k x'_k\right) \\
&= (x_0 + (e_{-2}, x_2)e_0, x'_0 + (e_{-2}, x'_2)e_0) + (x_{-2} + (e_0, x_0)e_{-2}, x'_2) \\
&+ (x_2, x'_{-2} + (e_0, x'_0)e_{-2}) + \sum_{k \neq -2, 0, 2} (x_{-k}, x'_k) + \sum_k (x_{-k}, x'_k) \\
&= (x_0, (e_{-2}, x'_2)e_0) + ((e_{-2}, x_2)e_0, x'_0) + ((e_0, x_0)e_{-2}, x'_2) + (x_2, (e_0, x'_0)e_{-2}) \\
&= (x_0, e_0)(e_{-2}, x'_2) + (e_{-2}, x_2)(e_0, x'_0) + (e_0, x_0)(e_{-2}, x'_2) + (x_2, e_{-2})(e_0, x'_0) = 0.
\end{aligned}$$

Thus  $B \in Sp(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $\sum_k x_k \mapsto (e_{-2}, x_2)e_0 \neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned}
& BA\left(\sum_k x_k\right) - AB\left(\sum_k x'_k\right) \\
&= \sum_{k \neq -2, 0} Ax_{k-2} + (Ax_{-4} + (e_0, Ax_{-2})e_{-2}) + (Ax_{-2} + (e_{-2}, Ax_0)e_0) \\
&- \sum_{k \neq -2, 0} Ax_k - (Ax_{-2} + (e_0, x_0)Ae_{-2}) - (Ax_0 + (e_{-2}, x_2)Ae_0) \\
&= -(Ae_0, x_{-2})e_{-2} - (Ae_{-2}, x_0)e_0 - (e_0, x_0)Ae_{-2} - (e_{-2}, x_2)Ae_0 = 0.
\end{aligned}$$

Thus  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We now assume that  $A : V_{-i} \rightarrow V_{-i+2}$  is injective for any  $i \geq 2$  and that for some even  $n > 0$ ,  $A^n : V_{-n} \rightarrow V_n$  is not an isomorphism. The kernel of this map is the radical of the symplectic form  $x, x' \mapsto (x, A^n x')$  on  $V_{-n}$  hence it has even codimension in  $V_{-n}$ ; but then it also has even dimension since  $\dim V_{-n}$  is even; since this kernel is non-zero it has dimension  $\geq 2$ . Thus we can find  $e_{-n}, f_{-n}$  linearly independent in  $V_{-n}$  such that  $A^n e_{-n} = 0, A^n f_{-n} = 0$ . For  $j \geq 0$  we set  $e_{2j-n} = A^j e_{-n}, f_{2j-n} = A^j f_{-n}$ . We have  $e_n = 0, f_n = 0$ . Also  $e_m, f_m$  are linearly independent in  $V_m$  if  $m \leq 0$  is even. For  $j \in [0, n]$  we have  $(e_{2j-n}, e_{n-2j}) = 0, (f_{2j-n}, f_{n-2j}) = 0, (e_{2j-n}, f_{n-2j}) = 0, (f_{2j-n}, e_{n-2j}) = 0$ . (The last of these equalities is equivalent to  $(A^j f_{-n}, A^{n-j} f_{-n}) = 0$  that is to  $(f_{-n}, A^n f_{-n}) = 0$  which follows from  $A^n f_{-n} = 0$ . The other three equalities are proved in a similar way.) Define  $B \in \text{End}(V)$  by

$$\begin{aligned} B\left(\sum_k x_k\right) &= \sum_{k \notin \{2h-n; h \in [0, n-1]\}} x_k \\ &+ \sum_{j \in [0, n-1]} (x_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2}) e_{2j-n} \\ &- (-1)^j (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n}). \end{aligned}$$

We have

$$\begin{aligned} &(B(\sum_k x_k), B(\sum_k x'_k)) - (\sum_k x_k, \sum_k x'_k) \\ &= \sum_{j \in [1, n-1]} (x_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2}) e_{2j-n} \\ &- (-1)^j (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n}, x'_{n-2j} + (-1)^{n-j} (f_{2j-n-2}, x'_{n-2j+2}) e_{n-2j} \\ &- (-1)^{n-j} (e_{2j-n-2}, x'_{n-2j+2}) f_{n-2j}) \\ &+ (x_{-n} + (f_{n-2}, x_{2-n}) e_{-n} - (e_{n-2}, x_{2-n}) f_{-n}, x'_n) \\ &+ (x_n, x'_{-n} + (f_{n-2}, x'_{2-n}) e_{-n} - (e_{n-2}, x'_{2-n}) f_{-n}) \end{aligned}$$



$$\begin{aligned}
& + \sum_{k \neq -n, 2-n, \dots, n-2, n} (x_k, x'_{-k}) - \sum_k (x_k, x'_{-k}) \\
& = \sum_{j \in [1, n]} (-1)^{n-j} (x_{2j-n}, (f_{2j-n-2}, x'_{n-2j+2}) e_{n-2j} \\
& \quad - (e_{2j-n-2}, x'_{n-2j+2}) f_{n-2j}) + \sum_{j \in [0, n-1]} (-1)^j (f_{n-2j-2}, x_{2j-n+2}) e_{2j-n} \\
& \quad - (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n}, x'_{n-2j}) \\
& = \sum_{j \in [1, n]} (-1)^{n-j} (x_{2j-n}, e_{n-2j}) (f_{2j-n-2}, x'_{n-2j+2}) \\
& \quad - \sum_{j \in [1, n]} (-1)^{n-j} (x_{2j-n}, f_{n-2j}) (e_{2j-n-2}, x'_{n-2j+2}) \\
& \quad + \sum_{j \in [0, n-1]} (-1)^j (f_{n-2j-2}, x_{2j-n+2}) (e_{2j-n}, x'_{n-2j}) \\
& \quad - \sum_{j \in [0, n-1]} (-1)^j (e_{n-2j-2}, x_{2j-n+2}) (f_{2j-n}, x'_{n-2j}) \\
& \quad = - \sum_{j \in [1, n]} (-1)^j (e_{n-2j}, x_{2j-n}) (f_{2j-n-2}, x'_{n-2j+2}) \\
& \quad + \sum_{j \in [1, n]} (-1)^j (e_{2j-n-2}, x'_{n-2j+2}) (f_{n-2j}, x_{2j-n}) \\
& \quad + \sum_{j \in [1, n]} (-1)^{j-1} (e_{2j-n-2}, x'_{n-2j+2}) (f_{n-2j}, x_{2j-n}) \\
& \quad - \sum_{j \in [1, n]} (-1)^{j-1} (e_{n-2j}, x_{2j-n}) (f_{2j-n-2}, x'_{n-2j+2}) = 0.
\end{aligned}$$

Thus  $B \in Sp(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $x_1 \mapsto (f_{-1}, x_1) e_{-1} - (e_{-1}, x_1) f_{-1}$  is  $\neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned}
& BA \left( \sum_k x_k \right) - AB \left( \sum_k x_k \right) \\
& = \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_{k-2} + \sum_{j \in [0, n-1]} (Ax_{-n-2+2j} \\
& \quad + (-1)^j (f_{n-2j-2}, Ax_{2j-n}) e_{2j-n} - (-1)^j (e_{n-2j-2}, Ax_{2j-n}) f_{2j-n}) \\
& \quad - \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_k \\
& \quad - \sum_{j \in [0, n-1]} (Ax_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2}) A e_{2j-n} \\
& \quad - (-1)^j (e_{n-2j-2}, x_{2j-n+2}) A f_{2j-n})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j \in [0, n-1]} (-1)^j ((f_{n-2j-2}, Ax_{2j-n})e_{2j-n} - (e_{n-2j-2}, Ax_{2j-n})f_{2j-n}) \\
&- \sum_{j \in [0, n-1]} (-1)^j ((f_{n-2j-2}, x_{2j-n+2})Ae_{2j-n} - (e_{n-2j-2}, x_{2j-n+2})Af_{2j-n}) \\
&= - \sum_{j \in [0, n-1]} (-1)^{j-1} ((f_{n-2j}, x_{2j-n})e_{2j-n} + (e_{n-2j}, x_{2j-n})f_{2j-n}) \\
&+ \sum_{j \in [0, n-1]} (-1)^j (-(f_{n-2j-2}, x_{2j-n+2})e_{2j-n+2} + (e_{n-2j-2}, x_{2j-n+2})f_{2j-n+2}) \\
&= \sum_{j \in [0, n-1]} (-1)^{j-1} ((f_{n-2j}, x_{2j-n})e_{2j-n} - (e_{n-2j}, x_{2j-n})f_{2j-n}) \\
&+ \sum_{j \in [1, n]} (-1)^{j-1} (-(f_{n-2j}, x_{2j-n})e_{2j-n} + (e_{n-2j}, x_{2j-n})f_{2j-n}) \\
&= -(f_n, x_{-n})e_{-n} - (e_n, x_{-n})f_{-n} + (-1)^n(f_{-n}, x_n)e_n - (-1)^n(e_{-n}, x_n)f_n = 0
\end{aligned}$$

since  $f_n = 0, e_n = 0$ . Thus  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We now assume that  $A : V_{-i} \rightarrow V_{-i+2}$  is injective for any  $i \geq 2$  and that for some odd  $n > 0$ ,  $A^n : V_{-n} \rightarrow V_n$  is not an isomorphism. We can find  $e_{-n} \in V_{-n} - \{0\}$  such that  $A^n e_{-n} = 0$ . For any  $j \geq 0$  we set  $e_{2j-n} = A^j e_{-n} \in V_{2j-n}$ . Note that  $e_n = 0$ . Also  $e_m \neq 0$  if  $m \leq 0$  is odd. We have  $(e_{2j-n}, e_{n-2j}) = 0$  for  $j \in [0, n]$ . Indeed we must show that  $(A^j e_{-n}, A^{n-j} e_{-n}) = 0$  that is,  $(e_{-n}, A^n e_{-n}) = 0$ . This follows from  $A^n e_{-n} = 0$ . Define  $B \in \text{End}(V)$  by

$$\begin{aligned}
&B\left(\sum_k x_k\right) \\
&= \sum_{k \notin \{2h-n; h \in [0, n-1]\}} x_k + \sum_{j \in [0, n-1]} (x_{2j-n} + (-1)^j (e_{n-2j-2}, x_{2j-n+2})e_{2j-n}).
\end{aligned}$$

We have

$$\begin{aligned}
&(B(\sum_k x_k), B(\sum_k x'_k)) - (\sum_k x_k, \sum_k x'_k) \\
&= \sum_{j \in [1, n-1]} (x_{2j-n} + (-1)^j (e_{n-2j-2}, x_{2j-n+2})e_{2j-n}, \\
&x'_{n-2j} + (-1)^{n-j} (e_{2j-n-2}, x'_{n-2j+2})e_{n-2j}) + (x_{-n} + (e_{n-2}, x_{2-n})e_{-n}, x'_n) \\
&+ (x_n, x'_{-n} + (e_{n-2}, x'_{2-n})e_{-n}) + \sum_{k \neq 0, 1, \dots, n-1} (x_{-k}, x'_k) - \sum_k (x_{-k}, x'_k)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j \in [1, n]} (-1)^{n-j} (x_{2j-n}, e_{n-2j}) (e_{2j-n-2}, x'_{n-2j+2}) \\
&+ \sum_{j \in [0, n-1]} (-1)^j ((e_{n-2j-2}, x_{2j-n+2}) (e_{2j-n}, x'_{n-2j})) \\
&= \sum_{j \in [1, n]} (-1)^j (e_{n-2j}, x_{2j-n}) (e_{2j-n-2}, x'_{n-2j+2}) \\
&+ \sum_{j \in [1, n]} (-1)^{j-1} ((e_{n-2j}, x_{2j-n}) (e_{2j-n-2}, x'_{n-2j+2})) = 0.
\end{aligned}$$

Thus  $B \in Sp(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $x_1 \mapsto \pm(e_{-1}, x_1)e_{-1}$  is  $\neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned}
BA\left(\sum_k x_k\right) - AB\left(\sum_k x'_k\right) &= \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_{k-2} \\
&+ \sum_{j \in [0, n-1]} (Ax_{2j-n-2} + (-1)^j (e_{n-2j-2}, Ax_{2j-n}) e_{2j-n}) \\
&- \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_k \\
&- \sum_{j \in [0, n-1]} (Ax_{2j-n} + (-1)^j (e_{n-2j-2}, x_{2j-n+2}) Ae_{2j-n}) \\
&= - \sum_{j \in [0, n-1]} (-1)^j (Ae_{n-2j-2}, x_{2j-n}) e_{2j-n} \\
&- \sum_{j \in [0, n-1]} (-1)^j (e_{n-2j-2}, x_{2j-n+2}) Ae_{2j-n} \\
&= - \sum_{j \in [0, n-1]} (-1)^j (e_{n-2j}, x_{2j-n}) e_{2j-n} \\
&+ \sum_{j \in [1, n]} (-1)^j (e_{n-2j}, x_{2j-n}) e_{2j-n} \\
&= -(e_n, x_{-n}) e_{-n} + (-1)^n (e_{-n}, x_n) e_n = 0.
\end{aligned}$$

Thus,  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We have shown that  $\mathfrak{s}(V)_2 - \mathfrak{s}(V)_2^0 \subset \mathfrak{g}_2^\delta - \mathfrak{g}_2^{\delta!}$ .

Conversely, let  $A \in \mathfrak{s}(V)_2^0$ . Let  $B \in G$  be such that  $AB = BA$ . It is enough to show that  $B \in G_{\geq 0}^\delta$ . Since  $A \in \text{End}(V)_2^0$  we see from the proof in 1.6 that  $B(V_{\geq i}) = V_{\geq i}$  for any  $i \in \mathbf{Z}$ . In particular,  $B \in G_{\geq 0}^\delta$ . This completes the proof.

**1.8.** We prove 1.5(a). Generally,  $x_k$  will denote an element of  $V_k$ . Let  $A \in \mathfrak{o}(V)_2 - \mathfrak{o}(V)_2^0$ .

Assume that  $A : V_{-i} \rightarrow V_{-i+2}$  is not injective for some  $i > 2$ . Then  $A : V_{i-2} \rightarrow V_i$  is not surjective and since  $\dim V_{i-2} \geq \dim V_i$ , we see that  $A : V_{i-2} \rightarrow V_i$  is not injective. We can find  $e_{-i} \in V_{-i} - \{0\}$  such that  $Ae_{-i} = 0$ . We can find  $e_{i-2} \in V_{i-2} - \{0\}$  such that  $Ae_{i-2} = 0$ . Define  $B \in \text{End}(V)$  by

$$B\left(\sum_k x_k\right) = \sum_{k \neq -i, i-2} x_k + (x_{-i} + (e_{i-2}, x_{-i+2})e_{-i}) + (x_{i-2} - (e_{-i}, x_i)e_{i-2}).$$

We have

$$\begin{aligned} QB\left(\sum_k x_k\right) - Q\left(\sum_k x_k\right) &= Q(x_0) + (x_{-i} + (e_{i-2}, x_{-i+2})e_{-i}, x_i) \\ &+ (x_{i-2} - (e_{-i}, x_i)e_{i-2}, x_{-i+2}) + \sum_{k>0, k \neq i-2, i} (x_{-k}, x_k) - Q(x_0) - \sum_{k>0} (x_{-k}, x_k) \\ &= ((e_{i-2}, x_{-i+2})e_{-i}, x_i) - ((e_{-i}, x_i)e_{i-2}, x_{-i+2}) \\ &= (e_{i-2}, x_{-i+2})(e_{-i}, x_i) - (e_{-i}, x_i)(e_{i-2}, x_{-i+2}) = 0. \end{aligned}$$

Hence  $B \in O(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $x_{-i+2} \mapsto (e_{i-2}, x_{-i+2})e_{-i}$  is  $\neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned} BA\left(\sum_k x_k\right) - AB\left(\sum_k x_k\right) &= \sum_{k \neq -i, i-2} Ax_{k-2} + (Ax_{-i-2} + (e_{i-2}, Ax_{-i})e_{-i}) + (Ax_{i-4} - (e_{-i}, Ax_{i-2})e_{i-2}) \\ &- \sum_{k \neq -i, i-2} Ax_k - (Ax_{-i} + (e_{i-2}, x_{-i+2})Ae_{-i}) - (Ax_{i-2} - (e_{-i}, x_i)Ae_{i-2}) \\ &= (e_{i-2}, Ax_{-i})e_{-i} - (e_{-i}, Ax_{i-2})e_{i-2} - (e_{i-2}, x_{-i+2})Ae_{-i} + (e_{-i}, x_i)Ae_{i-2} \\ &= -(Ae_{i-2}, x_{-i})e_{-i} + (Ae_{-i}, x_{i-2})e_{i-2} - (e_{i-2}, x_{-i+2})Ae_{-i} + (e_{-i}, x_i)Ae_{i-2} = 0. \blacksquare \end{aligned}$$

Hence  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

Next we assume that  $A : V_{-2} \rightarrow V_0$  is not injective that is,  $K := \ker(A : V_{-2} \rightarrow V_0) \neq 0$ . We can find  $e_{-2} \in V_{-2} - \{0\}$  such that  $Ae_{-2} = 0$ . Note that  $K' := \ker(A : V_0 \rightarrow V_2)$  is  $\neq 0$ . (If  $\dim V$  is odd and  $p = 2$  we have  $0 \neq \mathcal{R} \subset K'$ . If  $\dim V$  is even or if  $p \neq 2$  then  $A : V_0 \rightarrow V_2$  is the transpose of  $-A : V_{-2} \rightarrow V_0$  hence is not surjective. But  $\dim V_0 \geq \dim V_2$  hence again  $K' \neq 0$ .) We can find  $e_0 \in V_0 - \{0\}$  such that  $Ae_0 = 0$ . Let  $I' = \{x \in V_2; (x, K) = 0\}$ . Note that  $I' \neq V_2$ . Define a linear function  $f : V_2 \rightarrow \mathbf{k}$  by  $f(x_2) = (e_{-2}, x_2)\sqrt{Q(e_0)}$  where  $\sqrt{Q(e_0)}$  is a fixed square root of  $Q(e_0)$ . Now  $f = 0$  on  $I'$  since  $e_{-2} \in K$ . Hence  $f$  induces a linear function  $f' : V_2/I' \rightarrow \mathbf{k}$ . Note that  $K = (V_2/I')^*$  canonically. There is a unique linear function  $\gamma' : V_2/I' \rightarrow K$  such that  $(\gamma'(x_2), x'_2) = -f'(x_2)f'(x'_2)$  for all  $x_2, x'_2 \in V_2/I'$ . Hence there is a

unique linear function  $\gamma : V_2 \rightarrow V_{-2}$  such that  $(\gamma(x_2), x'_2) = -f(x_2)f(x'_2)$  for all  $x_2, x'_2$  in  $V_2$  and  $\gamma(V_2) \subset K$ ,  $\gamma(I') = 0$ . Hence  $A\gamma(V_2) = 0$ . Since  $AV_0 \subset I'$ , we have  $\gamma AV_0 = 0$ . Define  $B \in \text{End}(V)$  by

$$B\left(\sum_k x_k\right) = \sum_{k \neq -2, 0} x_k + (x_{-2} + (e_0, x_0)e_{-2} + \gamma(x_2)) + (x_0 - (e_{-2}, x_2)e_0).$$

We have

$$\begin{aligned} QB\left(\sum_k x_k\right) - Q\left(\sum_k x_k\right) &= Q(x_0) + (e_{-2}, x_2)^2 Q(e_0) - (e_0, x_0)(e_{-2}, x_2) + (x_{-2}, x_2) \\ &\quad + (e_0, x_0)(e_{-2}, x_2) + (\gamma(x_2), x_2) + \sum_{k>0, k \neq 2} (x_{-k}, x_k) - Q(x_0) - \sum_{k>0} (x_{-k}, x_k) \\ &= f(x_2)^2 + (\gamma(x_2), x_2) = 0. \end{aligned}$$

Thus  $B \in O(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $x_2 \mapsto (e_{-2}, x_2)e_0$  is  $\neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned} BA\left(\sum_k x_k\right) - AB\left(\sum_k x_k\right) &= \sum_{k \neq -2, 0} Ax_{k-2} + (Ax_{-4} + (e_0, Ax_{-2})e_{-2} + \gamma(Ax_0)) + (Ax_{-2} - (e_{-2}, Ax_0)e_0) \\ &\quad - \sum_{k \neq -2, 0} Ax_k - (Ax_{-2} + (e_0, x_0)Ae_{-2} + A\gamma(x_2)) - (Ax_0 - (e_{-2}, x_2)Ae_0) \\ &= -(Ae_0, x_{-2}, e_0)e_{-2} + \gamma(Ax_0) - (Ae_{-2}, x_0)e_0 - (e_0, x_0)Ae_{-2} - A\gamma(x_2) \\ &\quad + (e_{-2}, x_2)Ae_0 = \gamma(Ax_0) - A\gamma(x_2) = 0. \end{aligned}$$

Thus  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We now assume that  $A : V_{-i} \rightarrow V_{-i+2}$  is injective for any  $i \geq 2$  and that for some  $n > 0$ , and some  $\xi \in A^n(V_{-2n}) - \{0\}$  we have  $(\xi, A^n(V_{-2n})) = 0$ ,  $Q(\xi) = 0$ . We can write  $\xi = A^n e_{-2n}$  for a unique  $e_{-2n} \in V_{-2n} - \{0\}$ . For any  $j \geq 0$  we set  $e_{-2n+2j} = A^j e_{-2n} \in V_{-2n+2j}$ . Thus  $e_0 = \xi$  and  $(e_0, A^n V_{-2n}) = 0$ ,  $Q(e_0) = 0$ . We show that  $e_{2n} = 0$ . Indeed,  $(V_{-2n}, e_{2n}) = (V_{-2n}, A^n e_0) = \pm(A^n V_{-2n}, e_0) = 0$ . For  $j \in [0, 2n]$  we have:

$$(e_{-2n+2j}, e_{2n-2j}) = 0.$$

This follows from  $(A^j e_{-2n}, A^{2n-j} e_{-2n}) = \pm(A^{2n} e_{-2n}, e_{-2n}) = (e_{2n}, e_{-2n}) = 0$ .

Define  $B \in \text{End}(V)$  by

$$\begin{aligned} B\left(\sum_k x_k\right) &= \sum_{k \notin \{-2n+2h; h \in [0, 2n-1]\}} x_k \\ &\quad + \sum_{j \in [0, 2n-1]} (x_{-2n+2j} + (-1)^j (e_{2n-2j-2}, x_{-2n+2j+2})e_{-2n+2j}). \end{aligned}$$

We have

$$\begin{aligned}
& QB\left(\sum_k x_k\right) - Q\left(\sum_k x_k\right) \\
&= Q(x_0 + (e_{-2}, x_2)e_0) + \sum_{j \in [1, n-1]} (x_{-2n+2j} + (-1)^j (e_{2n-2j-2}, x_{-2n+2j+2})e_{-2n+2j}, \\
&\quad x_{2n-2j} + (-1)^{2n-j} (e_{-2n+2j-2}, x_{2n-2j+2})e_{2n-2j}) \\
&\quad + (x_{-2n} + (e_{2n-2}, x_{-2n+2})e_{-2n}, x_{2n}) \\
&\quad + \sum_{k > 0; k \neq 2, 4, \dots, 2n} (x_{-k}, x_k) - Q(x_0) - \sum_{k > 0} (x_{-k}, x_k) \\
&= Q(e_0) + (e_0, x_0)(e_{-2}, x_2) \\
&\quad + \sum_{j \in [1, n-1]} (-1)^j (e_{2n-2j}, x_{-2n+2j})(e_{-2n+2j-2}, x_{2n-2j+2}) \\
&\quad + \sum_{j \in [1, n-1]} (-1)^j (e_{2n-2j-2}, x_{-2n+2j+2})(e_{-2n+2j}, x_{2n-2j}) \\
&\quad + (e_{2n-2}, x_{-2n+2})(e_{-2n}, x_{2n}) - Q(x_0) - \sum_{k > 0} (x_{-k}, x_k) \\
&= \sum_{j \in [2, n-1]} (-1)^j (e_{2n-2j}, x_{-2n+2j})(e_{-2n+2j-2}, x_{2n-2j+2}) \\
&\quad + \sum_{j \in [1, n-2]} (-1)^j (e_{2n-2j-2}, x_{-2n+2j+2})(e_{-2n+2j}, x_{2n-2j}) \\
&= \sum_{j \in [2, n-1]} (-1)^j (e_{2n-2j}, x_{-2n+2j})(e_{-2n+2j-2}, x_{2n-2j+2}) \\
&\quad + \sum_{j \in [2, n-1]} (-1)^{j-1} (e_{2n-2j}, x_{-2n+2j})(e_{-2n+2j-2}, x_{2n-2j+2}) = 0.
\end{aligned}$$

Hence  $B \in SO(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $x_2 \mapsto$

$(e_{-2}, x_2)e_0$  is  $\neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned}
& BA\left(\sum_k x_k\right) - AB\left(\sum_k x_k\right) \\
&= \sum_{k \notin \{-2n+2h; h \in [0, 2n-1]\}} Ax_{k-2} \\
&+ \sum_{j \in [0, 2n-1]} (Ax_{-2n+2j-2} + (-1)^j (Ax_{-2n+2j}, e_{2n-2j-2})e_{-2n+2j}) \\
&- \sum_{k \notin \{-2n+2h; h \in [0, 2n-1]\}} Ax_k \\
&- \sum_{j \in [0, 2n-1]} (Ax_{-2n+2j} + (-1)^j (x_{-2n+2j+2}, e_{2n-2j-2})e_{-2n+2j+2}) \\
&= \sum_k Ax_k + \sum_{j \in [0, 2n-1]} (-1)^j (Ax_{-2n+2j}, e_{2n-2j-2})e_{-2n+2j}) \\
&- \sum_k Ax_k - \sum_{j \in [0, 2n-1]} (-1)^j (x_{-2n+2j+2}, e_{2n-2j-2})e_{-2n+2j+2} \\
&= - \sum_{j \in [0, 2n-1]} (-1)^j (x_{-2n+2j}, Ae_{2n-2j-2})e_{-2n+2j}) \\
&- \sum_{j \in [1, 2n]} (-1)^{j-1} (x_{-2n+2j}, e_{2n-2j})e_{-2n+2j} \\
&= -(x_{-2n}, e_{2n})e_{-2n} - (-1)^n (x_{2n}, e_{-2n})e_{2n} = 0
\end{aligned}$$

since  $e_{2n} = 0$ . Hence  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We now assume that  $A : V_{-i} \rightarrow V_{-i+2}$  is injective for any  $i \geq 2$  and that for some odd  $n > 0$ ,  $A^n : V_{-n} \rightarrow V_n$  is not an isomorphism. The kernel of this map is the radical of the symplectic form  $x, x' \mapsto (x, A^n x')$  on  $V_{-n}$  hence it has even codimension in  $V_{-n}$ ; but then it also has even dimension since  $\dim V_{-n}$  is even; since this kernel is non-zero it has dimension  $\geq 2$ . Thus we can find  $e_{-n}, f_{-n}$  linearly independent in  $V_{-n}$  such that  $A^n e_{-n} = 0, A^n f_{-n} = 0$ . For  $j \geq 0$  we set  $e_{2j-n} = A^j e_{-n}, f_{2j-n} = A^j f_{-n}$ . We have  $e_n = 0, f_n = 0$ . Also  $e_m, f_m$  are linearly independent in  $V_m$  if  $m < 0$  is odd. For  $j \in [0, n]$  we have  $(e_{2j-n}, e_{n-2j}) = 0, (f_{2j-n}, f_{n-2j}) = 0, (e_{2j-n}, f_{n-2j}) = 0, (f_{2j-n}, e_{n-2j}) = 0$ . (The last of these equalities is equivalent to  $(A^j f_{-n}, A^{n-j} f_{-n}) = 0$  that is, to  $(f_{-n}, A^n f_{-n}) = 0$  which follows from  $A^n f_{-n} = 0$ . The other three equalities are proved in a similar

way.) Define  $B \in \text{End}(V)$  by

$$\begin{aligned} B\left(\sum_k x_k\right) &= \sum_{k \notin \{2h-n; h \in [0, n-1]\}} x_k + \\ &\sum_{j \in [0, n-1]} (x_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2}) e_{2j-n} \\ &- (-1)^j (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n}). \end{aligned}$$

We have

$$\begin{aligned} &QB\left(\sum_k x_k\right) - Q\left(\sum_k x_k\right) \\ &= Q(x_0) + \sum_{j \in [1, (n-1)/2]} (x_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2}) e_{2j-n} \\ &- (-1)^j (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n}, \\ &x_{n-2j} + (-1)^{n-j} (f_{2j-n-2}, x_{n-2j+2}) e_{n-2j} - (-1)^{n-j} (e_{2j-n-2}, x_{n-2j+2}) f_{n-2j}) \\ &+ (x_{-n} + (f_{n-2}, x_{2-n}) e_{-n} - (e_{n-2}, x_{2-n}) f_{-n}, x_n) \\ &+ \sum_{k > 0; k \neq -n, 2-n, \dots, n-2} (x_{-k}, x_k) + Q(x_0) - \sum_{k > 0} (x_{-k}, x_k) - Q(x_0) \\ &= \sum_{j \in [1, (n-1)/2]} (x_{2j-n}, (-1)^{n-j} (f_{2j-n-2}, x_{n-2j+2}) e_{n-2j} \\ &- (-1)^{n-j} (e_{2j-n-2}, x_{n-2j+2}) f_{n-2j}) \\ &+ \sum_{j \in [0, (n-1)/2]} ((-1)^j (f_{n-2j-2}, x_{2j-n+2}) e_{2j-n} \\ &- (-1)^j (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n}, x_{n-2j}) \\ &= \sum_{j \in [1, (n-1)/2]} (-1)^{n-j} (e_{n-2j}, x_{2j-n}) (f_{2j-n-2}, x_{n-2j+2}) \\ &- \sum_{j \in [1, (n-1)/2]} (-1)^{n-j} (f_{n-2j}, x_{2j-n}) (e_{2j-n-2}, x_{n-2j+2}) \\ &- \sum_{j \in [0, (n-1)/2]} (-1)^j (f_{2j-n}, x_{n-2j}) (e_{n-2j-2}, x_{2j-n+2}) \\ &+ \sum_{j \in [0, (n-1)/2]} (-1)^j (e_{2j-n}, x_{n-2j}) (f_{n-2j-2}, x_{2j-n+2}) \end{aligned}$$



$$\begin{aligned}
&= \sum_{j \in [1, (n-1)/2]} (-1)^{n-j} (e_{n-2j}, x_{2j-n}) (f_{2j-n-2}, x_{n-2j+2}) \\
&- \sum_{j \in [1, (n-1)/2]} (-1)^{n-j} (f_{n-2j}, x_{2j-n}) (e_{2j-n-2}, x_{n-2j+2}) \\
&- \sum_{j \in [1, (n+1)/2]} (-1)^{j-1} (f_{2j-n-2}, x_{n+2-2j}) (e_{n-2j}, x_{2j-n}) \\
&+ \sum_{j \in [1, (n+1)/2]} (-1)^{j-1} (e_{2j-n-2}, x_{n+2-2j}) (f_{n-2j}, x_{2j-n}) \\
&= (-1)^{(n-1)/2} (e_{-1}, x_1) (f_{-1}, x_1) - (-1)^{(n-1)/2} (f_{-1}, x_1) (e_{-1}, x_1) = 0.
\end{aligned}$$

Thus,  $B \in O(V)$ . More precisely,  $B \in G_{<0}^\delta$ . We have  $B \neq 1$  since  $x_1 \mapsto \pm((f_{-1}, x_1)e_{-1} - (e_{-1}, x_1)f_{-1})$  is  $\neq 0$ . Hence  $B \notin G_{\geq 0}^\delta$ . We have

$$\begin{aligned}
&BA\left(\sum_k x_k\right) - AB\left(\sum_k x_k\right) = \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_{k-2} \\
&+ \sum_{j \in [0, n-1]} (Ax_{n-2+2j} + (-1)^j (f_{n-2j-2}, Ax_{2j-n}) e_{2j-n} - \\
&(-1)^j (e_{n-2j-2}, Ax_{2j-n}) f_{2j-n}) - \sum_{k \notin \{2h-n; h \in [0, n-1]\}} Ax_k \\
&- \sum_{j \in [0, n-1]} (Ax_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2}) A e_{2j-n} \\
&- (-1)^j (e_{n-2j-2}, x_{2j-n+2}) A f_{2j-n}) \\
&= \sum_{j \in [0, n-1]} (-1)^j ((f_{n-2j-2}, Ax_{2j-n}) e_{2j-n} - (e_{n-2j-2}, Ax_{2j-n}) f_{2j-n}) \\
&- \sum_{j \in [0, n-1]} (-1)^j ((f_{n-2j-2}, x_{2j-n+2}) A e_{2j-n} - (e_{n-2j-2}, x_{2j-n+2}) A f_{2j-n}) \\
&= - \sum_{j \in [0, n-1]} (-1)^{j-1} ((f_{n-2j}, x_{2j-n}) e_{2j-n} + (e_{n-2j}, x_{2j-n}) f_{2j-n}) \\
&+ \sum_{j \in [0, n-1]} (-1)^j (-(f_{n-2j-2}, x_{2j-n+2}) e_{2j-n+2} + (e_{n-2j-2}, x_{2j-n+2}) f_{2j-n+2}) \\
&= \sum_{j \in [0, n-1]} (-1)^{j-1} ((f_{n-2j}, x_{2j-n}) e_{2j-n} - (e_{n-2j}, x_{2j-n}) f_{2j-n}) \\
&+ \sum_{j \in [1, n]} (-1)^{j-1} (-(f_{n-2j}, x_{2j-n}) e_{2j-n} + (e_{n-2j}, x_{2j-n}) f_{2j-n}) \\
&= -(f_n, x_{-n}) e_{-n} + (e_n, x_{-n}) f_{-n} + (-1)^n (f_{-n}, x_n) e_n - (-1)^n (e_{-n}, x_n) f_n = 0
\end{aligned}$$

since  $f_n = 0$ ,  $e_n = 0$ . Thus  $AB = BA$ . We see that  $A \notin \mathfrak{g}_2^{\delta!}$ .

We have shown that  $\mathfrak{o}(V) - \mathfrak{o}(V)_2^0 \subset \mathfrak{g}_2^\delta - \mathfrak{g}_2^{\delta!}$ .

Conversely, let  $A \in \mathfrak{o}(V)_2^0$ . Let  $B \in G$  be such that  $AB = BA$ . It is enough to show that  $B \in G_{\geq 0}^\delta$ . We argue by induction on  $\dim V$ . If  $V = 0$  the result is clear. We now assume  $V \neq 0$ . Let  $m$  be the largest integer  $\geq 0$  such that  $V_m \neq 0$ . If  $m = 0$  we have  $G_{\geq 0}^\delta = G$  and the result is clear. Assume now that  $m \geq 1$ . If  $m$  is odd we have  $A^m V = V_m$ ,  $\ker(A^m : V \rightarrow V) = V_{\geq -m+1}$ . Since  $BA = AB$ , the image and kernel of  $A^m$  are  $B$ -stable. Hence  $B(V_m) = V_m$  and  $B(V_{\geq -m+1}) = V_{\geq -m+1}$ . Hence  $B$  induces an automorphism  $B' \in SO(V')$  where  $V' = V_{\geq -m+1}/V_m$ , a vector spaces with a nondegenerate quadratic form induced by  $Q$ . We have canonically  $V' = V_{-m+1} \oplus V_{-m+2} \oplus \dots \oplus V_{m-1}$  and  $\mathfrak{o}(V')_2, \mathfrak{o}(V')_2^0$  are defined in terms of this ( $\mathfrak{o}$ -good) grading. Now  $A$  induces an element  $A' \in \mathfrak{o}(V')_2^0$  and we have  $B'A' = A'B'$ . By the induction hypothesis, for any  $i \in [-m+1, m-1]$ , the subspace  $V_i + V_{i+1} + \dots + V_{m-1}$  of  $V'$  is  $B'$ -stable. Hence the subspace  $V_{\geq i}$  of  $V$  is  $B$ -stable. We see that  $B \in G_{\geq 0}^\delta$ . Next we assume that  $m$  is even. We have  $V_{\geq -m+1} = \{x \in V; A^m(x) = 0, Q(A^{m/2}x) = 0\}$  (we use that  $A \in \mathfrak{o}(V)_2^0$ ). Since  $B$  commutes with  $A$  and preserves  $Q$  we see that  $B$  preserves the subspace  $\{x \in V; A^m(x) = 0, Q(A^{m/2}x) = 0\}$  hence  $B(V_{\geq -m+1}) = V_{\geq -m+1}$ . We have  $V_m = \{x \in V; (x, V_{\geq -m+1}) = 0, Q(x) = 0\}$ . Since  $B$  preserves the subspace  $V_{\geq -m+1}$  and  $B$  preserves  $Q$  and  $(,)$  we see that  $B(V_m) = V_m$ . Hence  $B$  induces an automorphism  $B' \in SO(V')$  where  $V' = (V_{\geq -m+1})/V_m$ , a vector space with a nondegenerate quadratic form induced by  $Q$ . We have canonically  $V' = V_{-m+1} \oplus V_{-m+2} \oplus \dots \oplus V_{m-1}$  and  $\mathfrak{o}(V')_2, \mathfrak{o}(V')_2^0$  are defined in terms of this ( $\mathfrak{o}$ -good) grading. Now  $A$  induces an element  $A' \in \mathfrak{o}(V')_2^0$  and we have  $B'A' = A'B'$ . By the induction hypothesis, for any  $i \in [-m+1, m-1]$ , the subspace  $V_i + V_{i+1} + \dots + V_{m-1}$  of  $V'$  is  $B'$ -stable. Hence the subspace  $V_{\geq i}$  of  $V$  is  $B$ -stable. We see that  $B \in G_{\geq 0}^\delta$ . This completes the proof.

**1.9.** Let  $\delta \in \mathfrak{D}_G$ . We describe the set of  $G_0^\delta$ -orbits of  $\mathfrak{g}_2^{\delta!}$  (see 1.2) in the cases considered in 1.3-1.5. If  $G, \mathfrak{g}$  are as in 1.3 or as in 1.4 (with  $p \neq 2$ ) or 1.5 (with  $p \neq 2$ ) then, using 1.3(a), 1.4(a), 1.5(a), we see that  $G_0^\delta$  acts transitively on  $\mathfrak{g}_2^{\delta!}$ . If  $V, G, \mathfrak{g}$  are as in 1.4 (with  $p = 2$ ) then the set of  $G_0^\delta$ -orbits of  $\mathfrak{g}_2^{\delta!} = \mathfrak{s}(V)_2^0$  is a set (with cardinal a power of 2) described in [L2, p.478]. In the rest of this subsection we assume that  $V, Q, (,), G, \mathfrak{g}$  are as in 1.5 and  $p = 2$ . Let  $(V_i)$  be the  $\mathfrak{o}$ -good grading of  $V$  corresponding to  $\delta$ . For any  $n \geq 0$  let  $d_n = \dim V_{-2n}$ . Let  $M = \{n \geq 0; d_n = \text{odd}\}$ . If  $M = \emptyset$  then  $G_0^\delta$  acts transitively on  $\mathfrak{g}_2^{\delta!} = \mathfrak{o}(V)_2^0$ . Now assume that  $M \neq \emptyset$ . We write the elements of  $M$  in increasing order  $n_0 < n_1 < \dots < n_t$ . Let  $X$  be the set of all functions  $f : \{1, 2, \dots, t\} \rightarrow \{0, 1\}$  such that:

- (i)  $f(i) = 1$  if  $n_i - n_{i-1} \geq 2$ ;
- (ii)  $f(i) = 0$  if  $n_i - n_{i-1} = 1$  and  $d_{n_{i-1}} = d_{n_i}$ .

Note that  $|X| = 2^\alpha$  where  $\alpha = |\{i \in [1, t]; n_{i-1} = n_i - 1, d_{n_{i-1}} > d_{n_i}\}|$ .

For any  $A \in \mathfrak{o}(V)_2^0$  and any  $n \in M$  let  $L_n^A$  be the radical of the restriction of  $(,)$  to  $K_n^A := A^n(V_{-2n})$  (a line).

Note that: if  $0 \leq s < q < r$ ,  $s \in M, q \notin M, r \in M$  then  $L_s^A \neq L_r^A$ . (Indeed, let  $x \in L_s^A - 0$ . Then  $(x, K_q^A) = 0$  hence  $x \notin K_q^A$  hence  $x \notin K_r^A$  and  $x \notin L_r^A$ .) If  $0 \leq s < q < r$ ,  $s \in M, q \in M, r \in M$  and  $L_s^A = L_r^A$  then  $L_s^A = L_q^A = L_r^A$ . (Indeed, let  $x \in L_s^A - 0$ . Then  $(x, K_q^A) = 0$ . But  $x \in L_r^A \subset K_r^A \subset K_q^A$  hence  $x \in L_q^A$ . Thus  $L_s^A = L_q^A$ .) If  $s \in M, q \in M$ ,  $K_s^A = K_q^A$  then clearly  $L_s^A = L_q^A$ .

We define  $f_A : \{1, 2, \dots, t\} \rightarrow \{0, 1\}$  by  $f_A(i) = 0$  if  $L_{n_i}^A = L_{n_{i-1}}^A$ ,  $f_A(i) = 1$  if  $L_{n_i}^A \neq L_{n_{i-1}}^A$ . From the previous paragraph we see that  $f_A \in X$  and that for any  $i, j$  in  $[0, t]$ ,  $f_A$  determines whether  $L_{n_i}^A, L_{n_j}^A$  are equal or not.

For any  $f \in X$  we set  ${}^f\mathfrak{o}(V)_2^0 = \{A \in \mathfrak{o}(V)_2^0; f_A = f\}$ . Then the subsets  ${}^f\mathfrak{o}(V)_2^0$  ( $f \in X$ ) are exactly the orbits of  $G_0^\delta$  on  $\mathfrak{g}_2^{\delta!} = \mathfrak{o}(V)_2^0$ .

## 2. THE PIECES IN THE UNIQUOTENT VARIETY OF $G$

**2.1.** Given  $\delta, \delta'$  in  $\mathfrak{D}_G$  we write  $\delta \sim \delta'$  if for any  $i \in \mathbf{N}$  we have  $G_{\geq i}^\delta = G_{\geq i}^{\delta'}$  or equivalently  $\mathfrak{g}_{\geq i}^\delta = \mathfrak{g}_{\geq i}^{\delta'}$ . This is an equivalence relation on  $\mathfrak{D}_G$ . Let  $D_G$  be the set of equivalence classes. The conjugation  $G$ -action on  $\mathfrak{D}_G$  induces a  $G$ -action on  $D_G$ . If  $\Delta \in D_G$ ,  $i \in \mathbf{Z}$ , we can write  $G_{\geq i}^\Delta, \mathfrak{g}_{\geq i}^\Delta$  instead of  $G_{\geq i}^\delta, \mathfrak{g}_{\geq i}^\delta$  where  $\delta \in \Delta$ . If  $\Delta \in D_G$ , we have an action of  $G_{\geq 0}^\Delta$  on  $\Delta$  given by  $g : \delta \mapsto \delta'$  where  $\delta'(a) = g\delta(a)g^{-1}$  for all  $a \in \mathbf{k}^*$ . We show:

(a) *the conjugation action of  $G_{\geq 0}^\Delta$  on the set of pairs  $(\delta, T)$  where  $\delta \in \Delta$  and  $T$  is a maximal torus of  $G_{\geq 0}^\Delta$  containing  $\delta(\mathbf{k}^*)$  is transitive; hence the conjugation action of  $G_{\geq 0}^\Delta$  on  $\Delta$  is transitive.*

Let  $(\delta, T), (\delta', T')$  be two pairs as above. Since  $T, T'$  are conjugate in  $G_{\geq 0}^\Delta$  we can assume that  $T' = T$ . It is enough to show that in this case we have  $\delta = \delta'$ . For any root  $\alpha : T \rightarrow \mathbf{k}^*$  of  $G$  with respect to  $T$  we set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; \text{Ad}(t)x = \alpha(t)x \quad \forall t \in T\}.$$

For any root  $\alpha$  define  $\langle \delta, \alpha \rangle \in \mathbf{Z}, \langle \delta', \alpha \rangle \in \mathbf{Z}$  by  $\alpha(\delta(a)) = a^{\langle \delta, \alpha \rangle}, \alpha(\delta'(a)) = a^{\langle \delta', \alpha \rangle}$  for all  $a \in \mathbf{k}^*$ . For  $i > 0$  we have

$$\bigoplus_{\alpha; \langle \delta, \alpha \rangle = i} \mathfrak{g}_\alpha = \mathfrak{g}_{\geq i}^\delta = \mathfrak{g}_{\geq i}^{\delta'} = \bigoplus_{\alpha; \langle \delta', \alpha \rangle = i} \mathfrak{g}_\alpha.$$

Since each  $\mathfrak{g}_\alpha$  is 1-dimensional and the sum  $\sum_\alpha \mathfrak{g}_\alpha$  is direct it follows that

$$\{\alpha; \langle \delta, \alpha \rangle = i\} = \{\alpha; \langle \delta', \alpha \rangle = i\}$$

for any  $i > 0$ . But then we automatically have  $\{\alpha; \langle \delta, \alpha \rangle = i\} = \{\alpha; \langle \delta', \alpha \rangle = i\}$  for any  $i \in \mathbf{Z}$ . It follows that  $\langle \delta, \alpha \rangle = \langle \delta', \alpha \rangle$  for any root  $\alpha$ . Define  $\mu \in \text{Hom}(\mathbf{k}^*, T)$  by  $\mu(a) = \delta'(a)\delta(a)^{-1}$  for any  $a \in \mathbf{k}^*$ . Then  $\langle \mu, \alpha \rangle = 0$  for any root  $\alpha$ . Thus  $\mu(\mathbf{k}^*)$  is contained in the centre of  $G$ . Since  $\delta(\mathbf{k}^*), \delta'(\mathbf{k}^*)$  are contained in  $G^{\text{der}}$ , we have  $\mu(\mathbf{k}^*) \subset G^{\text{der}}$ . Thus  $\mu(\mathbf{k}^*)$  is contained in the centre of  $G^{\text{der}}$ , a finite group. Since  $\mu(\mathbf{k}^*)$  is connected we have  $\mu(\mathbf{k}^*) = \{1\}$  hence  $\delta' = \delta$ . This proves (a).

From (a) we see that

(b) *the obvious map  $\mathfrak{D}_G \rightarrow D_G$  induces a bijection  $G \backslash \mathfrak{D}_G \rightarrow G \backslash D_G$  on the sets of  $G$ -orbits.*

In the remainder of this subsection we assume that

(c)  *$p > 1$ ,  $\mathbf{k}$  is an algebraic closure of the field  $\mathbf{F}_p$  with  $p$  elements and we are given a split  $\mathbf{F}_p$ -rational structure on  $G$  with Frobenius map  $F : G \rightarrow G$ .*

This induces a split  $\mathbf{F}_p$ -rational structure on  $\mathfrak{g}$ . For any  $\delta \in \text{Hom}(\mathbf{k}^*, G)$  we define  ${}^F\delta : \mathbf{k}^* \rightarrow G$  by  $({}^F\delta)(z) = F(\delta(z^{p^{-1}}))$  for  $z \in \mathbf{k}^*$ . We show that for some  $g \in G$  we have  ${}^F\delta(z) = g\delta(z)g^{-1}$  for all  $z$ . Let  $T$  be a maximal torus of  $G$  such that  $F(t) = t^p$  for all  $t \in T$ . We can find  $g_1 \in G$  such that  $g_1\delta(\mathbf{k}^*)g_1^{-1} \subset T$ . Then for  $z \in \mathbf{k}^*$  we have  $g_1\delta(z)g_1^{-1} \in T$  hence  $F(g_1\delta(z)g_1^{-1}) = (g_1\delta(z)g_1^{-1})^p = g_1\delta(z^p)g_1^{-1} = F(g_1){}^F\delta(z^p)F(g_1^{-1})$  hence  ${}^F\delta(z) = g\delta(z)g^{-1}$  where  $g = F(g_1)^{-1}g_1$ , as claimed. In particular, if  $\delta \in \mathfrak{D}_G$  then  ${}^F\delta \in \mathfrak{D}_G$  and  ${}^F\delta$  is in the same  $G$ -orbit as  $\delta$ . From the definitions we see that if  $\delta \in \mathfrak{D}_G$  and  $i \in \mathbf{N}$  then  $\mathfrak{g}_{\geq i}^{({}^F\delta)} = F(\mathfrak{g}_{\geq i}^\delta)$  and  $G_{\geq i}^{({}^F\delta)} = F(G_{\geq i}^\delta)$ . In particular if  $\delta, \delta'$  in  $\mathfrak{D}_G$  satisfy  $\delta \sim \delta'$  then  ${}^F\delta \sim {}^F\delta'$ . Thus the permutation  $\delta \mapsto {}^F\delta$  of  $\mathfrak{D}_G$  induces a permutation  $\Delta \mapsto {}^F\Delta$  of  $D_G$ . This permutation maps each  $G$ -orbit in  $D_G$  into itself.

**2.2.** Let  $\Delta \in D_G$ ,  $i > 0$ . Let  $(\delta, T)$  be such that  $\delta \in \Delta$  and  $T$  is a maximal torus of  $G_{\geq 0}^\Delta$  containing  $\delta(\mathbf{k}^*)$ . We will define an isomorphism of algebraic groups

$$\Phi_{\delta, T} : \mathfrak{g}_{\geq i}^\Delta / \mathfrak{g}_{\geq i+1}^\Delta \xrightarrow{\sim} G_{\geq i}^\Delta / G_{\geq i+1}^\Delta.$$

For any root  $\alpha : T \rightarrow \mathbf{k}^*$  we define the root subspaces  $\mathfrak{g}_\alpha$  as in the proof of 2.1(a). Let  $G_\alpha$  be the root subgroup of  $G$  corresponding to  $\mathfrak{g}_\alpha$ . For any  $\alpha$  we can find an isomorphism of algebraic groups  $h_\alpha : \mathbf{k} \xrightarrow{\sim} G_\alpha$ ; it is unique up to composing with multiplication by a nonzero scalar on  $\mathbf{k}$ ; by passage to Lie algebras,  $h_\alpha$  gives rise to an isomorphism of vector spaces  $h'_\alpha : \mathbf{k} \xrightarrow{\sim} \mathfrak{g}_\alpha$ . For any  $j \geq 0$  let  $R_j$  be the set of roots  $\alpha$  such that  $\langle \alpha, \delta \rangle = j$  (notation as in the proof of 2.1(a)). Then

$$\phi : \mathbf{k}^{R_i} \rightarrow \mathfrak{g}_{\geq i}^\Delta / \mathfrak{g}_{\geq i+1}^\Delta, \quad (c_\alpha) \mapsto \sum_{\alpha \in R_i} h'_\alpha(c_\alpha) \mod \mathfrak{g}_{\geq i+1}^\Delta,$$

$$\psi : \mathbf{k}^{R_i} \rightarrow G_{\geq i}^\Delta / G_{\geq i+1}^\Delta, \quad (c_\alpha) \mapsto \prod_{\alpha \in R_i} h_\alpha(c_\alpha) \mod G_{\geq i+1}^\Delta$$

are isomorphisms of algebraic groups. (The last product is independent of the order of the factors up to  $\mod G_{\geq i+1}^\Delta$ , by the Chevalley commutator formula.) We set  $\Phi_{\delta, T} = \psi\phi^{-1}$ . Clearly  $\Phi_{\delta, T}$  is independent of the choice of the  $h_\alpha$ . Now  $G_{\geq 0}^\Delta$  acts by conjugation on  $G_{\geq i}^\Delta / G_{\geq i+1}^\Delta$  and this induces an Ad-action of  $G_{\geq 0}^\Delta$  on  $\mathfrak{g}_{\geq i}^\Delta / \mathfrak{g}_{\geq i+1}^\Delta$ . From the definitions we see that for any  $g \in G_{\geq 0}^\Delta$  and any  $\xi \in \mathfrak{g}_{\geq i}^\Delta / \mathfrak{g}_{\geq i+1}^\Delta$  we have  $\Phi_{g\delta g^{-1}, gTg^{-1}}(\xi) = g\Phi_{\delta, T}(\text{Ad}(g^{-1})\xi)g^{-1}$ . Let  $\alpha \in R_i, \beta \in R_j, j \geq 0$  and let  $c, c' \in$

**k.** If  $j > 0$  we have  $h_\beta(c)h_\alpha(c')h_\beta(c)^{-1} = h_\alpha(c') \pmod{G_{\geq i+1}^\Delta}$ ,  $\text{Ad}(h_\beta(c))h'_\alpha(c') = h'_\alpha(c') \pmod{\mathfrak{g}_{\geq i+1}^\Delta}$ . If  $j = 0$  we have (by the Chevalley commutator formula)

$$h_\beta(c)h_\alpha(c')h_\beta(c)^{-1} = h_\alpha(c') \prod_{i' > 0} h_{i'\beta+\alpha}(m_{i'}c^{i'}c') \pmod{G_{\geq i+1}^\Delta},$$

where  $m_{i'} \in \mathbf{Z}$  are such that

$$\text{Ad}(h_\beta(c))h'_\alpha(c') = h'_\alpha(c') + \sum_{i' > 0} h'_{i'\beta+\alpha}(m_{i'}c^{i'}c').$$

From these formulas we see that for any  $\xi \in \mathfrak{g}_{\geq i}^\Delta/\mathfrak{g}_{\geq i+1}^\Delta$  we have  $\Phi_{\delta,T}(\text{Ad}(g)\xi) = g\Phi_{\delta,T}(\xi)g^{-1}$  whenever  $g = h_\beta(c)$  with  $\beta \in \cup_{j \geq 0} R_j$ ,  $c \in \mathbf{k}$ . The same holds when  $g \in T$  and even for any  $g$  in  $G_{\geq 0}^\Delta$  since this group is generated by  $T$  and by the  $h_\beta(c)$  as above. We see that for any  $g \in G_{\geq 0}^\Delta$  and any  $\xi \in \mathfrak{g}_{\geq i}^\Delta/\mathfrak{g}_{\geq i+1}^\Delta$  we have  $g\Phi_{\delta,T}(\text{Ad}(g^{-1})x)g^{-1} = \Phi_{\delta,T}(\xi)$ , hence  $\Phi_{g\delta g^{-1}, gTg^{-1}}(\xi) = \Phi_{\delta,T}(\xi)$ . Using this and 2.1(a) we see that  $\Phi_{\delta,T}$  is independent of the choice of  $\delta, T$ . Hence it can be denoted by  $\Phi_\Delta$ . We can summarize the results above as follows.

(a) For any  $\Delta \in D_G$  and any  $i > 0$  there is a canonical  $G_{\geq 0}^\Delta$ -equivariant isomorphism of algebraic groups  $\Phi_\Delta : \mathfrak{g}_{\geq i}^\Delta/\mathfrak{g}_{\geq i+1}^\Delta \xrightarrow{\sim} G_{\geq i}^\Delta/G_{\geq i+1}^\Delta$ .

**2.3.** Let  $\Delta \in D_G$ . For any  $\delta \in \Delta$  the subset  $\mathfrak{g}_2^{\delta!}$  of  $\mathfrak{g}_2^\delta$  can be viewed as a subset  $\Sigma^\delta$  of  $\mathfrak{g}_{\geq 2}^\Delta/\mathfrak{g}_{\geq 3}^\Delta$  via the obvious isomorphism  $\mathfrak{g}_2^\delta \xrightarrow{\sim} \mathfrak{g}_{\geq 2}^\Delta/\mathfrak{g}_{\geq 3}^\Delta$ . From the definitions, for any  $g \in G_{\geq 0}^\Delta$  we have  $\Sigma^{g\delta g^{-1}} = \text{Ad}(g)\Sigma^\delta$ . Here we use the Ad-action of  $G_{\geq 0}^\Delta$  on  $\mathfrak{g}_{\geq 2}^\Delta/\mathfrak{g}_{\geq 3}^\Delta$ . This action factors through an action of  $G_{\geq 0}^\Delta/G_{\geq 1}^\Delta = G_0^\delta$ . Hence if we write  $g = g_0g'$  where  $g_0 \in G_0^\delta$ ,  $g' \in G_{\geq 1}^\Delta$  we have  $\text{Ad}(g)(\Sigma^\delta) = \text{Ad}(g_0)\Sigma^\delta = \Sigma^\delta$  (the last equality follows from  $\text{Ad}(g_0)\mathfrak{g}_2^{\delta!} = \mathfrak{g}_2^{\delta!}$ ). Thus we have  $\Sigma^{g\delta g^{-1}} = \Sigma^\delta$ . Using 2.1(a) we deduce that  $\Sigma^\delta$  is independent of the choice of  $\delta$  in  $\Delta$ ; we will denote it by  $\Sigma^\Delta$ . Note that  $\Sigma^\Delta$  is a subset of  $\mathfrak{g}_{\geq 2}^\Delta/\mathfrak{g}_{\geq 3}^\Delta$  stable under the action of  $G_{\geq 0}^\Delta$ .

Let  $\mathfrak{S}^\Delta \subset G_{\geq 2}^\Delta$  be the inverse image of  $\Sigma^\Delta$  under the composition  $G_{\geq 2}^\Delta \rightarrow G_{\geq 2}^\Delta/G_{\geq 3}^\Delta \xrightarrow{\Phi_\Delta^{-1}} \mathfrak{g}_{\geq 2}^\Delta/\mathfrak{g}_{\geq 3}^\Delta$  where the first map is the obvious one. Now  $\mathfrak{S}^\Delta$  is stable under the conjugation action of  $G_{\geq 0}^\Delta$  on  $G_{\geq 2}^\Delta$  and  $u \mapsto u$  is a map

$$(a) \quad \Psi_G : \sqcup_{\Delta \in D_G} \mathfrak{S}^\Delta \rightarrow \mathcal{U}_G, u \mapsto u.$$

**Theorem 2.4.** Assume that  $\tilde{G}^{der}$  (see 1.1) is a product of almost simple groups of type  $A, B, C, D$ . Then  $\Psi_G$  is a bijection.

The general case reduces easily to the case where  $G$  is almost simple of type  $A, B, C$  or  $D$ . Moreover we can assume that  $G$  is one of the groups  $GL(V)$ ,  $Sp(V)$ ,  $SO(V)$  in 1.3-1.5. The proof in these cases will be given in 2.5-2.10. We expect that the theorem holds without restriction on  $G$ .

We now discuss some applications of the theorem. Let  $\mathfrak{A}_G$  be the set of  $G$ -orbits on  $D_G$ . Using 2.1(b) and the definitions we see that  $\mathfrak{A}_G = \mathfrak{A}_{G'}$  where  $G'$  is as in 1.1. In particular  $\mathfrak{A}_G$  is a finite set which depends only on the type of  $G$ , not on

**k.** For any  $\mathcal{O} \in \mathfrak{A}_G$  we consider the set

$$Z_{\mathcal{O}} = \sqcup_{\Delta \in \mathcal{O}} \Sigma^\Delta.$$

Note that  $G$  acts naturally on  $Z_{\mathcal{O}}$  (this action induces the conjugation action of

$G$  on  $\mathcal{O}$ ) and for any  $\Delta \in \mathcal{O}$  the obvious map  $G_{\geq 0}^\Delta \backslash \Sigma^\Delta \mapsto G \backslash Z_{\mathcal{O}}$  is a bijection denoted by  $\omega_\Delta \leftrightarrow \omega$ . (We use the fact the stabilizer in  $G$  of an element  $\Delta \in \mathcal{O}$  is equal to  $G_{\geq 0}^\Delta$ .) Since  $G_{\geq 0}^\Delta \backslash \Sigma^\Delta$  is a finite set whose cardinal is a power of 2 (see 1.9) we see that  $G \backslash Z_{\mathcal{O}}$  is a finite set whose cardinal is a power of 2. For any  $\Delta \in D_G$  and  $\omega \in G \backslash Z_{\mathcal{O}}$  let  $\mathfrak{S}_\omega^\Delta$  be the inverse image of  $\omega_\Delta$  under the map  $\mathfrak{S}^\Delta \rightarrow \Sigma^\Delta$  in 2.3; we have a partition  $\mathfrak{S}^\Delta = \sqcup_{\omega \in G \backslash Z_{\mathcal{O}}} \mathfrak{S}_\omega^\Delta$ . We set

$$\begin{aligned} \mathcal{U}_G^\mathcal{O} &= \Psi_G(\sqcup_{\Delta \in \mathcal{O}} \mathfrak{S}^\Delta), \\ \mathcal{U}_G^{\mathcal{O}, \omega} &= \Psi_G(\sqcup_{\Delta \in \mathcal{O}} \mathfrak{S}_\omega^\Delta), \quad (\omega \in G \backslash Z_{\mathcal{O}}). \end{aligned}$$

The subsets  $\mathcal{U}_G^\mathcal{O}$  are called the *pieces* of  $\mathcal{U}_G$ . They form a partition of  $\mathcal{U}_G$  into subsets (which are unions of  $G$ -orbits) indexed by  $\mathfrak{A}_G = \mathfrak{A}_{G'}$ . The subsets  $\mathcal{U}_G^{\mathcal{O}, \omega}$  are called the *subpieces* of  $\mathcal{U}_G$ . When  $\omega$  varies in  $G \backslash Z_{\mathcal{O}}$  the subpieces  $\mathcal{U}_G^{\mathcal{O}, \omega}$  form a partition of a piece  $\mathcal{U}_G^\mathcal{O}$  into subsets (which are unions of  $G$ -orbits); the number of these subsets is a power of 2.

If  $G$  is as in 1.3 then each piece of  $\mathcal{U}_G$  is a single  $G$ -orbit. If  $G$  is as in 1.4 ( $p = 2$ ) then each subpiece of  $\mathcal{U}_G$  is a single  $G$ -orbit.. If  $G, V$  are as in 1.5 ( $p = 2$ ,  $\dim V = 9$ ) then there exist a subpiece of  $\mathcal{U}_G$  which is a union of two  $G$ -orbits.

In the remainder of this subsection we assume that 2.1(c) holds. Then for each  $\mathcal{O} \in \mathfrak{A}_G$  and  $n \geq 1$ , the piece  $\mathcal{U}_G^\mathcal{O}$  is  $F$ -stable and according to [L2], [L3], the number  $|(\mathcal{U}_G^\mathcal{O})^{F^n}|$  is a polynomial in  $p^n$  with integer coefficients independent of  $p, n$ . From the definitions we have

$$(a) \quad |(\mathcal{U}_G^\mathcal{O})^{F^n}| = |\mathcal{O}^{F^n}| \cdot |(G_{\geq 3}^\Delta)^{F^n}| \cdot |(\Sigma^\Delta)^{F^n}|$$

where  $\Delta$  is any point of  $\mathcal{O}^F$  (Note that  $\mathcal{O}$  is  $F$ -stable by 2.1.)

**2.5.** Let  $V, G, \mathfrak{g}$  be as in 1.3. A filtration of  $V$  is a collection of subspaces  $V_* = (V_{\geq a})_{a \in \mathbf{Z}}$  of  $V$  such that  $V_{\geq a+1} \subset V_{\geq a}$  for all  $a$ ,  $V_{\geq a} = 0$  for some  $a$ ,  $V_{\geq a} = V$  for some  $a$ . If a filtration as above is given, we set  $\text{gr}_a(V_*) = V_{\geq a}/V_{\geq a+1}$  for any  $a$  and  $\text{gr}(V_*) = \oplus_a \text{gr}_a(V_*)$ . Let  $\mathfrak{F}(V)$  be the set of filtrations  $V_* = (V_{\geq a})_{a \in \mathbf{Z}}$  of  $V$  such that the grading  $(\text{gr}_a(V_*))$  of  $\text{gr}(V_*)$  is good (see 1.3) or equivalently such that there exists a good grading  $(V_i)$  of  $V$  with  $V_{\geq a} = \oplus_{i; i \geq a} V_i$  for all  $i$ .

If  $\delta, \delta' \in \mathfrak{D}_G$  correspond to the good gradings  $(V_i), (V'_i)$  of  $V$  then  $\delta \sim \delta'$  if and only if  $\oplus_{i; i \geq a} V_i = \oplus_{i; i \geq a} V'_i$  for all  $a \in \mathbf{Z}$ . Setting  $V_{\geq a} = \oplus_{i; i \geq a} V_i$  we see that  $(V_{\geq a}) \in \mathfrak{F}(V)$  and that  $(\sim \text{-equivalence class of } \delta) \mapsto (V_{\geq a})$  is a bijection  $D_G \xrightarrow{\sim} \mathfrak{F}(V)$ .

For any  $V_* = (V_{\geq a}) \in \mathfrak{F}(V)$  let  $\xi(V_*)$  be the set of all  $A \in \text{End}(V)$  such that  $A(V_{\geq a}) \subset V_{\geq a+2}$  for any  $a \in \mathbf{Z}$  and such that the map  $\bar{A} \in \text{End}(\text{gr}(V_*))_2$  induced by  $A$  belongs to  $\text{End}(\text{gr}(V_*))_2^0$ . Note that  $\xi(V_*) \subset \mathcal{N}_{\mathfrak{g}}$ . Using 1.3(a) we see that in our case the following statement is equivalent to 2.4:

$$(a) \quad \text{the map } \sqcup_{V_* \in \mathfrak{F}(V)} \xi(V_*) \rightarrow \mathcal{N}_{\mathfrak{g}}, \quad A \mapsto A, \text{ is a bijection.}$$

Let  $A \mapsto V_*^A$  be the map  $\mathcal{N}_{\mathfrak{g}} \rightarrow \mathfrak{F}(V)$  defined in [L2, 2.3]. Then  $A \mapsto (A, V_*^A)$  is a well defined map  $\mathcal{N}_{\mathfrak{g}} \rightarrow \sqcup_{V_* \in \mathfrak{F}(V)} \xi(V_*)$  which, by [L2, 2.4] is an inverse of the map (a).

**2.6.** Let  $V, (, ), G, \mathfrak{g} = \mathfrak{s}(V)$  be as in 1.4. A filtration  $V_* = (V_{\geq a})_{a \in \mathbf{Z}}$  of  $V$  is said to be self-dual if  $\{x \in V; (x, V_{\geq a}) = 0\} = V_{\geq 1-a}$  for any  $a$ . If this is so,

the associated vector space  $\text{gr}(V_*)$  has a unique symplectic form  $(\cdot, \cdot)_0$  such that  $(\text{gr}_a(V_*), \text{gr}_{a'}(V_*))_0 = 0$  if  $a + a' \neq 0$  and, for any  $x \in \text{gr}_a(V)$ ,  $y \in \text{gr}_{-a}(V)$ , we have  $(x, y)_0 = (\dot{x}, \dot{y})$  where  $\dot{x} \in V_{\geq a}$ ,  $\dot{y} \in V_{\geq -a}$  are representatives of  $x, y$ . Note that  $(\cdot, \cdot)_0$  is nondegenerate. Let  $\mathfrak{F}_s(V)$  be the set of self-dual filtrations  $V_* = (V_{\geq a})_{a \in \mathbf{Z}}$  of  $V$  such that the grading  $(\text{gr}_a(V_*))$  of  $\text{gr}(V_*)$  is  $s$ -good (see 1.4) or equivalently such that there exists an  $s$ -good grading  $(V_i)$  of  $V$  with  $V_{\geq a} = \bigoplus_{i; i \geq a} V_i$  for all  $i$ .

If  $\delta, \delta' \in \mathfrak{D}_G$  correspond to the  $s$ -good gradings  $(V_i), (V'_i)$  of  $V$  then  $\delta \sim \delta'$  if and only if  $\bigoplus_{i; i \geq a} V_i = \bigoplus_{i; i \geq a} V'_i$  for all  $a \in \mathbf{Z}$ . Setting  $V_{\geq a} = \bigoplus_{i; i \geq a} V_i$  we see that  $(V_{\geq a}) \in \mathfrak{F}_s(V)$  and that  $(\sim \text{--equivalence class of } \delta) \mapsto (V_{\geq a})$  is a bijection  $D_G \xrightarrow{\sim} \mathfrak{F}_s(V)$ .

Let  $\mathcal{M}(V)$  be the set of all nilpotent elements  $A \in \text{End}(V)$  such that  $(Ax, y) + (x, Ay) + (Ax, Ay) = 0$  for all  $x, y$  in  $V$  (or equivalently such that  $1 + A \in \text{Sp}(V)$ ). For any  $V_* = (V_{\geq a}) \in \mathfrak{F}_s(V)$  let  $\tilde{\xi}(V_*)$  be the set of all  $A \in \mathcal{M}(V)$  such that  $A(V_{\geq a}) \subset V_{\geq a+2}$  for any  $a \in \mathbf{Z}$  and such that the map  $\bar{A} \in \text{End}(\text{gr}(V_*))_2$  induced by  $A$  belongs to  $\text{End}(\text{gr}(V_*))_2^0$ . Using 1.4(a) we see that in our case the following statement is equivalent to 2.4:

(a) *the map  $\sqcup_{V_* \in \mathfrak{F}_s(V)} \tilde{\xi}(V_*) \rightarrow \mathcal{M}(V)$ ,  $A \mapsto A$ , is a bijection.*

If  $A \in \mathcal{M}(V)$  then  $V_*^A \in \mathfrak{F}(V)$  (see 2.5) is self-dual (see [L2, 3.2(c)]) and the map  $\bar{A} \in \text{End}(\text{gr}(V_*))_2$  induced by  $A$  is in  $\text{End}(\text{gr}(V_*))_2^0$  and is skew-adjoint with respect to  $(\cdot, \cdot)_0$  (see [L2, 3.2(d)]); this implies that  $\dim \text{gr}_a(V_*)$  is even when  $a$  is even, hence  $V_*^A \in \mathfrak{F}_s(V)$ . Then  $A \mapsto (A, V_*^A)$  is a well defined map  $\mathcal{M}(V) \rightarrow \sqcup_{V_* \in \mathfrak{F}_s(V)} \tilde{\xi}(V_*)$  which, by [L2, 2.4] is an inverse of the map (a).

**2.7.** Let  $V, Q, (\cdot, \cdot), G = SO(V), \mathfrak{g} = \mathfrak{o}(V)$  be as in 1.5. A filtration  $V_* = (V_{\geq a})_{a \in \mathbf{Z}}$  of  $V$  is said to be a  $Q$ -filtration if for any  $a \geq 1$  we have  $Q|_{V_{\geq a}} = 0$  and  $\{x \in V; (x, V_{\geq a}) = 0\} = V_{\geq 1-a}$ . Then  $Q$  induces (as in [L3, 1.5]) a nondegenerate quadratic form  $\bar{Q} : \text{gr} V_* \rightarrow \mathbf{k}$ .

Let  $\mathfrak{F}_o(V)$  be the set of all  $Q$ -filtrations  $V_* = (V_{\geq a})_{a \in \mathbf{Z}}$  of  $V$  such that the grading  $(\text{gr}_a(V_*))$  of  $\text{gr}(V_*)$  is  $o$ -good (see 1.5) or equivalently such that there exists an  $o$ -good grading  $(V_i)$  of  $V$  with  $V_{\geq a} = \bigoplus_{i; i \geq a} V_i$  for all  $i$ .

If  $\delta, \delta' \in \mathfrak{D}_G$  correspond to the  $o$ -good gradings  $(V_i), (V'_i)$  of  $V$  then  $\delta \sim \delta'$  if and only if  $\bigoplus_{i; i \geq a} V_i = \bigoplus_{i; i \geq a} V'_i$  for all  $a \in \mathbf{Z}$ . Setting  $V_{\geq a} = \bigoplus_{i; i \geq a} V_i$  we see that  $(V_{\geq a}) \in \mathfrak{F}_o(V)$  and that  $(\sim \text{--equivalence class of } \delta) \mapsto (V_{\geq a})$  is a bijection  $D_G \xrightarrow{\sim} \mathfrak{F}_o(V)$ . Let

$$\mathcal{M}(V) = \{A \in \text{End}(V); A \text{ nilpotent}, 1 + A \in SO(V)\}.$$

For any  $V_* = (V_{\geq a}) \in \mathfrak{F}_o(V)$  let  $\eta(V_*)$  be the set of all  $A \in \mathcal{M}(V)$  such that  $A(V_{\geq a}) \subset V_{\geq a+2}$  for any  $a \in \mathbf{Z}$  and such that the map  $\bar{A} \in \text{End}(\text{gr}(V_*))_2$  induced by  $A$  belongs to  $\mathfrak{o}(\text{gr}(V_*))_2^0$  (the last set is defined in terms of the grading  $(\text{gr}_a(V_*))$  and the quadratic form on  $\text{gr}(V_*)$  induced by  $Q$ ). Using 1.3(a) we see that in our case the following statement is equivalent to 2.4:

(a) *the map  $\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*) \rightarrow \mathcal{M}(V)$ ,  $A \mapsto A$ , is a bijection.*

Assume first that  $p \neq 2$ . If  $A \in \mathcal{M}(V)$  then  $V_*^A \in \mathfrak{F}(V)$  (see 2.5) is in  $\mathfrak{F}_o(V)$  (see

[L3, 3.3]). Then  $A \mapsto (A, V_*^A)$  is a well defined map  $\mathcal{M}(V) \rightarrow \sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*)$  which, by [L2, 2.4] is an inverse of the map (a).

Next we assume that  $p = 2$ . If  $A \in \mathcal{M}(V)$  we define a filtration  $V_A^* = (V_A^{\geq a})$  of  $V$  as in [L3, 2.5]. (Note that in general  $V_A^*$  is not equal to  $V_*^A$  of 2.5.) From [L3, 2.5(a)] we see that  $V_A^*$  is a  $Q$ -filtration and from [L3, 2.9] we see that  $V_*^A \in \mathfrak{F}_o(V)$  and  $A \in \eta(V_*^A)$ . Then  $A \mapsto (A, V_A^*)$  is a well defined map  $\mathcal{M}(V) \rightarrow \sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*)$  which, by [L3, 2.10] is an inverse of the map (a). (An alternative proof of (a) is given in 2.9-2.10.)

**2.8.** Let  $V, Q, (.,.), G = SO(V)$  be as in 1.5. In the remainder of this section we assume that  $p = 2$ .

For any nonzero nilpotent element  $T \in \text{End}(V)$  let  $e = e_T$  be the smallest integer  $\geq 2$  such that  $T^e = 0$ ; let  $f = f_T$  be the smallest integer  $\geq 1$  s.t.  $QT^f = 0$ . We associate to  $T$  a subset  $H_T$  of  $V$  as follows.

- (i)  $H_T = \{x \in V; T^{e-1}x = 0\}$  if  $e \geq 2f$ ,
- (ii)  $H_T = \{x \in V; T^{e-1}x = 0, QT^{f-1}x = 0\}$  if  $e = 2f - 1$ ,
- (iii)  $H_T = \{x \in V; QT^{f-1}x = 0\}$  if  $e < 2f - 1$ .

We fix an  $o$ -good grading  $V = \oplus_i V_i$  such that  $V \neq V_0$ . For  $a \in \mathbf{Z}$  let  $V_{\geq a} = V_a + V_{a+1} + \dots$ . Let  $m \geq 1$  be the largest integer such that  $V_m \neq 0$ . Let  $A \in \mathfrak{o}(V)_2^0$ . Note that  $A \neq 0$  and  $A$  is nilpotent hence  $\bar{e} := e_A, \bar{f} := f_A, H_A$  are defined.

From the definition of  $\mathfrak{o}(V)_2^0$  we see that:

(a) *If  $m$  is odd then  $\bar{e} = 2\bar{f} = m+1$ . If  $m$  is even then either  $\bar{e} = 2\bar{f} - 1 = m+1$  or  $\bar{e} < 2\bar{f} - 1 = m+1$ . In any case,  $V_{\geq -m+1} = H_A$ .*

Let  $T \in \text{End}(V)$  be such that  $C := T - A$  satisfies  $C(V_i) \subset V_{\geq i+3}$  for any  $i$ . Then  $T \neq 0, T(V_i) \subset V_{\geq i+2}$  for any  $i$  hence  $T$  is nilpotent so that  $e = e_T, f = f_T, H_T$  are defined. We shall prove successively the statements (b)-(j).

(b) *If  $2n > m$  then  $QT^n = 0, QA^n = 0$ . Hence if  $m$  is odd then  $f \leq (m+1)/2, \bar{f} \leq (m+1)/2$ ; if  $m$  is even then  $f \leq (m+2)/2, \text{bof} \leq (m+2)/2$ .*

To show that  $QT^n = 0$  it is enough to show that  $Q(T^n x) = 0$  whenever  $x \in V_i, i \geq -m$  and  $(T^n x, T^n x') = 0$  whenever  $x \in V_i, x' \in V_j, i, j \geq -m$ . This follows from  $T^n x \in V_{\geq 1}$  and  $Q|_{V_{\geq 1}} = 0, (V_{\geq 1}, V_{\geq 1}) = 0$ . The equality  $QA^n = 0$  is proved in the same way.

The following result is immediate.

(c) *Let  $P_n \in \text{End}(V)$  be a product of  $n$  factors of which at least one is  $C$  and the remaining ones are  $A$ . Then  $P_n(V_i) \subset V_{\geq i+2n+1}$  for all  $i$ . Hence if  $n \geq m$  then  $P_n = 0$ .*

(d) *If  $n \geq m$  then  $T^n = A^n$ .*

Indeed,  $T^n = A^n + \sum P_n$  where each term  $P_n$  is as in (c). Hence the result follows from (c).

(e) *If  $\bar{e} \geq 2\bar{f} - 1$  then  $T^{\bar{e}-1} = A^{\bar{e}-1} \neq 0, T^{\bar{e}} = A^{\bar{e}} = 0$ ; hence  $e = \bar{e}$ . Moreover if  $\bar{e} = 2\bar{f}$  then  $H_T = H_A$ .*

In our case we have  $\bar{e} = m+1$  (see (a)) hence the first sentence follows from (d).



If  $\bar{e} = 2\bar{f}$  then  $m$  is odd and  $e = \bar{e} = 2\bar{f} = m + 1 \geq 2f$  (see (b)) hence  $e \geq 2f$  and

$$\begin{aligned} H_T &= \{x \in V; T^{e-1}x = 0\} = \{x \in V; T^{\bar{e}-1}x = 0\} \\ &= \{x \in V; A^{\bar{e}-1}x = 0\} = H_A. \end{aligned}$$

(f) If  $2n \geq m$ ,  $P_n$  is as in (c) and  $P'_n$  is like  $P_n$  then  $QP_n(V) = 0$ ,  $(P_n(V), P'_n(V)) = 0$ ,  $(A^n(V), P_n(V)) = 0$ . ■

By (c) we have  $P_n V_i \subset V_{\geq 1}$  if  $i \geq -m$  hence  $P_n(V) \subset V_{\geq 1}$ . Also,  $A^n V \subset V_{\geq 0}$ . It is then enough to use:  $Q|_{V_{\geq 1}} = 0$ ,  $(V_{\geq 1}, V_{\geq 0}) = 0$ .

(g) If  $2n \geq m$  then  $QT^n = QA^n$ .

We have  $T^n = A^n + \sum P_n$  where each term  $P_n$  is as in (c). Hence for  $x \in V$  we have  $QT^n x = QA^n x + \sum QP_n x + \sum (A^n, P_n x) + \sum (P_n x, P'_n x)$  with  $P'_n$  like  $P_n$  and we use (f).

(h) If  $\bar{e} < 2\bar{f}$  then  $QT^{\bar{f}-1} = QA^{\bar{f}-1} \neq 0$ ,  $QT^{\bar{f}} = QA^{\bar{f}} = 0$ ; hence  $f = \bar{f}$ .

In this case we have  $2(\bar{f} - 1) = m$ ,  $2\bar{f} > m$ . Hence result follows from (g).

(i) If  $\bar{e} < 2\bar{f} - 1$  then  $e < 2f - 1$ .

In this case we have  $f = \bar{f} = (m + 2)/2$ . We must show that  $e < m + 1$ . By (d) we have  $T^m = A^m$ . Since  $\bar{e} \leq m$  (see (a)) we have  $A^m = 0$  hence  $T^m = 0$  and  $e < m + 1$ .

(j) If  $2\bar{f} = \bar{e}$  then  $e = \bar{e} = m + 1 \geq f/2$ . If  $2\bar{f} - 1 = \bar{e}$  then  $e = \bar{e} = m + 1$ ,  $f = \bar{f}$  hence  $2f - 1 = e$ . If  $2\bar{f} - 1 > \bar{e}$  then  $f = \bar{f} = (m + 2)/2$  and  $2f - 1 > e$ . In each case we have  $H_T = H_A$  hence  $H_T = V_{\geq -m+1}$  and  $m = \max(e - 1, 2f - 2)$ .

**2.9.** We give an alternative proof of the injectivity of the map 2.7(a). We argue by induction on  $\dim V$ . If  $\dim V \leq 1$ , the result is trivial. Assume now that  $\dim V \geq 2$ . Let  $T \in \mathcal{M}(V)$  and let  $V_* = (V_{\geq a})$ ,  $\tilde{V}_* = (\tilde{V}_{\geq a})$  be two filtrations in  $\mathfrak{F}_o(V)$  such that  $T \in \eta(V_*)$  and  $T \in \eta(\tilde{V}_*)$ . We must show that  $V_* = \tilde{V}_*$ . Let  $\bar{T} \in \text{End}(\text{gr}V_*)_2$ ,  $\bar{T}_1 \in \text{End}(\text{gr}\tilde{V}_*)_2$  be the endomorphisms induced by  $T$ . If  $\bar{T} = 0$  then, using the fact that  $T \in \eta'(V_*)$  we see that  $\text{gr}_a(V_*) = 0$  for  $a \neq 0$  hence  $V_{\geq 1} = 0$ ,  $V_{\geq 0} = V$ ; since  $TV = T(V_{\geq 0}) \subset V_{\geq 2} = 0$  we see that  $T = 0$ , hence  $\bar{T}_1 = 0$  and  $\tilde{V}_{\geq 1} = 0$ ,  $\tilde{V}_{\geq 0} = V$ ; thus  $V_* = \tilde{V}_*$  as desired. Similarly, if  $\bar{T}_1 = 0$  then  $V_* = \tilde{V}_*$  as desired. Thus we can assume that  $\bar{T} \neq 0$ ,  $\bar{T}_1 \neq 0$ . Hence  $\text{gr}_a(V_*) \neq 0$  for some  $a \neq 0$  and  $\text{gr}_a(\tilde{V}_*) \neq 0$  for some  $a \neq 0$ . Let  $m \geq 1$  be the largest integer such that  $\text{gr}_m(V_*) \neq 0$ . Let  $\tilde{m} \geq 1$  be the largest integer such that  $\text{gr}_{\tilde{m}}(\tilde{V}_*) \neq 0$ . Using 2.8(j) we see that  $V_{\geq -m+1} = H_T = \tilde{V}_{\geq -\tilde{m}+1}$ ,  $m = \max(e_T - 1, 2f_T - 2) = \tilde{m}$ . It follows that  $m = \tilde{m}$  and  $V_{\geq -m+1} = \tilde{V}_{\geq -m+1}$ . Since  $V_*$  is a  $Q$ -filtration we have  $V_{\geq m} = \{x \in V; (x, V_{\geq -m+1}) = 0; Q(x) = 0\}$ , see [L3, 1.4(b)]. Similarly we have  $\tilde{V}_{\geq m} = \{x \in V; (x, \tilde{V}_{\geq -m+1}) = 0; Q(x) = 0\}$ . Hence  $V_{\geq m} = \tilde{V}_{\geq m}$ . Let  $V' = V_{\geq -m+1}/V_{\geq m} = \tilde{V}_{\geq -m+1}/\tilde{V}_{\geq m}$ . Note that  $V'$  has a natural nondegenerate quadratic form induced by  $Q$ . We set  $V'_{\geq a} = \text{image of } V_{\geq a} \text{ under } V_{\geq -m+1} \rightarrow V'$  (if  $a \geq -m + 1$ ),  $V'_{\geq a} = 0$  (if  $a < -m + 1$ ). We set  $\tilde{V}'_{\geq a} = \text{image of } \tilde{V}_{\geq a} \text{ under } \tilde{V}_{\geq -m+1} \rightarrow V'$  (if  $a \geq -m + 1$ ),  $\tilde{V}'_{\geq a} = 0$  (if  $a < -m + 1$ ). Then  $V'_* = (V'_{\geq a})$ ,  $\tilde{V}'_* = (\tilde{V}'_{\geq a})$  are filtrations in  $\mathfrak{F}_o(V')$ . Also  $T$

induces an element  $T' \in \mathcal{M}(V')$  and we have  $T' \in \eta(V'_*)$ ,  $T' \in \eta(\tilde{V}'_*)$ . Note also that  $\dim V' < \dim V$ . By the induction hypothesis we have  $V'_* = \tilde{V}'_*$ . It follows that  $V_{\geq a} = \tilde{V}_{\geq a}$  for any  $a \geq -m + 1$ . If  $a < -m + 1$  we have  $V_{\geq a} = \tilde{V}_{\geq a} = V$ . Hence  $V_* = \tilde{V}_*$ , as desired. Thus the map 2.7(a) is injective.

**2.10.** We give an alternative proof of the surjectivity of the map 2.7(a). By a standard argument we can assume that  $\mathbf{k}$  is an algebraic closure of the field  $\mathbf{F}_2$  with 2 elements. We can also assume that  $\dim V \geq 2$ . We choose an  $\mathbf{F}_2$ -rational structure on  $V$  such that  $Q$  is defined and split over  $\mathbf{F}_2$ . Then the Frobenius map relative to the  $\mathbf{F}_2$ -structure acts naturally and compatibly on the source and target of the map 2.7(a). We denote each of these actions by  $F$ . It is enough to show that for any  $n \geq 1$  the map  $\alpha_n : (\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*))^{F^n} \rightarrow \mathcal{M}(V)^{F^n}$ ,  $A \mapsto A$  is a bijection. (Here  $()^{F^n}$  is the fixed point set of  $F^n$ .) Since  $\alpha_n$  is injective (see 2.9) it is enough to show that  $|\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*)^{F^n}| = |\mathcal{M}(V)^{F^n}|$ . By [St, 15.1] we have  $|\mathcal{M}(V)^{F^n}| = 2^{nr}$  where  $r$  is the number of roots of  $G$ . It is enough to show that

$$(a) \sum_{V_* \in \mathfrak{F}_o(V)^{F^n}} |\eta(V_*)^{F^n}| = 2^{nr}.$$

Now the left hand side of (a) makes sense when  $\mathbf{k}$  is replaced by an algebraic closure of the prime field with  $p'$  elements where  $p'$  is any prime number; moreover this more general expression is a polynomial in  $p'^n$  with integer coefficients independent of  $p', n$  (see [L3, 1.5(e)]). When  $p' \neq 2$ , this more general expression is equal to  $p'^{nr}$  since 2.7(a) is already known to be a bijection in this case. But then (a) follows from this more general equality by specializing  $p'^n$  (viewed as indeterminate) to  $2^n$ .

**2.11.** Assume that  $G$  is as in 2.4 and that  $\Delta, \tilde{\Delta} \in D_G$  are conjugate under  $G$ . We show:

$$(a) \text{ if } u \in \mathfrak{S}^\Delta, u \in G_{\geq 2}^{\tilde{\Delta}} \text{ for some } u \text{ then } \Delta = \tilde{\Delta}.$$

We can assume that  $G$  is as in 1.3, 1.4 or 1.5.

Assume first that  $G, \mathfrak{g}, V$  are as in 1.3. We prove (a) by induction on  $\dim V$ . If  $V = 0$  the result is trivial. We now assume that  $V \neq 0$ . Let  $V_* = (V_{\geq a})$ ,  $\tilde{V}_* = (\tilde{V}_{\geq a})$  be the objects of  $\mathfrak{F}(V)$  corresponding to  $\Delta, \tilde{\Delta}$ . Let  $m$  be the largest integer  $\geq 0$  such that  $\text{gr}_m V_* \neq 0$ . If  $m = 0$  then  $V_a = V$  for  $a \leq m$  and  $V_a = 0$  for  $a > m$ . Since  $\tilde{V}_*, V_*$  are  $G$ -conjugate we have  $\tilde{V}_* = V_*$ . Thus we may assume that  $m \geq 1$ . We have  $V_{\geq -m} = V$ ,  $V_{\geq m+1} = 0$ . Since  $\tilde{V}_*, V_*$  are  $G$ -conjugate we see that  $\tilde{V}_{\geq -m} = V$ ,  $\tilde{V}_{\geq m+1} = 0$ . We set  $x = u - 1$ . From  $x \in \xi(V_*)$  we deduce  $V_{\geq -m+1} = \ker(x^m : V \rightarrow V)$ ,  $V_{\geq m} = x^m V$ . From  $x(\tilde{V}_{\geq a}) \subset \tilde{V}_{\geq a+2}$  for all  $a$  we deduce  $x^m(\tilde{V}_{\geq -m+1}) \subset \tilde{V}_{\geq -m+1+2m} = 0$ ,  $x^m(V) = x^m(\tilde{V}_{\geq -m}) \subset \tilde{V}_{\geq -m+2m} = \tilde{V}_{\geq m}$ . Thus,  $\tilde{V}_{\geq -m+1} \subset \ker(x^m : V \rightarrow V)$  and  $\tilde{V}_{\geq -m+1} \subset V_{\geq -m+1}$ ; moreover,  $V_{\geq m} \subset \tilde{V}_{\geq m}$ . Since  $\tilde{V}_*, V_*$  are  $G$ -conjugate we have  $\dim \tilde{V}_{\geq -m+1} = \dim V_{\geq -m+1}$ ,  $\dim \tilde{V}_{\geq m} = \dim V_{\geq m}$  hence  $\tilde{V}_{\geq -m+1} = V_{\geq -m+1}$ ,  $\tilde{V}_{\geq m} = V_{\geq m}$ . Let  $V' = V_{\geq -m+1}/V_{\geq m} = \tilde{V}_{\geq -m+1}/\tilde{V}_{\geq m}$ . Now  $V_*, \tilde{V}_*$  give rise in an obvious way to two elements  $V'_*, \tilde{V}'_*$  of  $\mathfrak{F}(V')$  and  $u$  gives rise to a unipotent element  $u' \in GL(V')$

such that  $u' - 1 \in \xi(V'_*)$  and  $(u' - 1)(\tilde{V}'_{\geq a}) \subset \tilde{V}'_{\geq a+2}$  for all  $a$ . Since  $\dim V' < \dim V$ , the induction hypothesis shows that  $V'_* = \tilde{V}'_*$ . It follows that  $V_* = \tilde{V}_*$ . This proves (a) in our case.

Assume next that  $G, \mathfrak{g}, V, (, )$  are as in 1.4. In this case the proof is essentially the same as in the case of 1.3. The same applies in the case where  $G, \mathfrak{g}, V, Q, (, )$  are as in 1.5 and  $p \neq 2$ .

In the remainder of this subsection we assume that  $G, V, Q, (, )$  are as in 1.5 and  $p = 2$ . We prove (a) by induction on  $\dim V$ . If  $V = 0$  the result is trivial. We now assume that  $V \neq 0$ . Let  $V_* = (V_{\geq a})$ ,  $\tilde{V}_* = (\tilde{V}_{\geq a})$  be the objects of  $\mathfrak{F}_o(V)$  corresponding to  $\Delta, \tilde{\Delta}$ . Let  $m$  be the largest integer  $\geq 0$  such that  $\text{gr}_m V_* \neq 0$ . If  $m = 0$  then  $V_a = V$  for  $a \leq m$  and  $V_a = 0$  for  $a > m$ . Since  $\tilde{V}_*, V_*$  are  $G$ -conjugate we have  $\tilde{V}_* = V_*$ . Thus we may assume that  $m \geq 1$ . We have  $V_{\geq -m} = V$ . Since  $\tilde{V}_*, V_*$  are  $G$ -conjugate we see that  $\tilde{V}_{\geq -m} = V$ . Let  $x = u - 1$ . From  $x(\tilde{V}_{\geq a}) \subset \tilde{V}_{\geq a+2}$  for all  $a$  we deduce  $x^m(\tilde{V}_{\geq -m+1}) \subset \tilde{V}_{\geq -m+1+2m} = 0$ ; moreover if  $m$  is even we have  $Qx^{m/2}(\tilde{V}_{\geq -m+1}) \subset Q\tilde{V}_{\geq -m+1+m} = Q\tilde{V}_{\geq 1} = 0$ .

Thus,  $\tilde{V}_{\geq -m+1} \subset \{v \in V; x^m v = 0\}$  and if  $m$  is even,  $\tilde{V}_{\geq -m+1} \subset \{v \in V; Qx^{m/2}v = 0\}$ . Recall from 2.8(j) that  $m = \max(e_x - 1, 2f_x - 2)$ . If  $e_x \geq 2f_x$  we have  $m = e_x - 1$  and

$$\tilde{V}_{\geq -m+1} \subset \{v \in V; x^{e_x-1}v = 0\} = H_x = V_{\geq -m+1}.$$

If  $e_x < 2f_x - 1$  we have  $m = 2f_x - 2$  and

$$\tilde{V}_{\geq -m+1} \subset \{v \in V; Qx^{f_x-1}v = 0\} = H_x = V_{\geq -m+1}.$$

If  $e_x = 2f_x - 1$  we have  $m = 2f_x - 2 = e_x - 1$  and

$$\tilde{V}_{\geq -m+1} \subset \{v \in V; x^{e_x-1}v = 0\} \cap \{v \in V; Qx^{f_x-1}v = 0\} = H_x = V_{\geq -m+1}.$$

Thus in any case we have  $\tilde{V}_{\geq -m+1} \subset V_{\geq -m+1}$ . It follows that  $\tilde{V}_{\geq m} = V_{\geq m}$ . Let  $V' = V_{\geq -m+1}/V_{\geq m} = \tilde{V}_{\geq -m+1}/\tilde{V}_{\geq m}$ . Note that  $Q$  induces naturally a nondegenerate quadratic form on  $V'$ . Also  $V_*, \tilde{V}_*$  give rise in an obvious way to two elements  $V'_*, \tilde{V}'_*$  of  $\mathfrak{F}_o(V')$  and  $u$  gives rise to a unipotent element  $u' \in SO(V')$  such that  $(u' - 1) \in \eta'(V'_*)$  and  $(u' - 1)(\tilde{V}'_{\geq a}) \subset \tilde{V}'_{\geq a+2}$  for all  $a$ . Since  $\dim V' < \dim V$ , the induction hypothesis shows that  $V'_* = \tilde{V}'_*$ . It follows that  $V_* = \tilde{V}_*$ . This proves (a) in our case.

#### APPENDIX: THE PIECES IN THE NILPOTENT VARIETY OF $\mathfrak{g}$

by *G. Lusztig and T. Xue*

**A.1.** For any  $\Delta \in D_G$  let  $\mathfrak{s}^\Delta \subset \mathfrak{g}_{\geq 2}^\Delta$  be the inverse image of  $\Sigma^\Delta$  (see 2.3) under the obvious map  $\mathfrak{g}_{\geq 2}^\Delta \rightarrow \mathfrak{g}_{\geq 2}^\Delta/\mathfrak{g}_{\geq 3}^\Delta$ . Now  $\mathfrak{s}^\Delta$  is stable under the  $\text{Ad}$ -action of  $G_{\geq 0}^\Delta$  on  $\mathfrak{g}_{\geq 2}^\Delta$  and  $x \mapsto x$  is a map

$$\Psi_{\mathfrak{g}} : \sqcup_{\Delta \in D_G} \mathfrak{s}^\Delta \rightarrow \mathcal{N}_{\mathfrak{g}}.$$

**Theorem A.2.** Assume that  $\tilde{G}^{\text{der}}$  (see 1.1) is a product of almost simple groups of type  $A, B, C, D$ . Then  $\Psi_{\mathfrak{g}}$  is a bijection.

The general case reduces easily to the case where  $G$  is almost simple of type  $A, B, C$  or  $D$ . Moreover we can assume that  $G$  is one of the groups  $GL(V)$ ,  $Sp(V)$ ,  $SO(V)$  in 1.3-1.5. The proof in these cases will be given in A.3-A.4. We expect that the theorem holds without restriction on  $G$ .

**A.3.** If  $V, G, \mathfrak{g}$  are as in 1.3, the proof of A.2 is exactly as in 2.5. Now assume that  $V, (, ), G, \mathfrak{g} = \mathfrak{s}(V)$  are as in 1.4. For any  $V_* = (V_{\geq a}) \in \mathfrak{F}_s(V)$  let  $\tilde{\xi}'(V_*)$  be the set of all  $A \in \mathfrak{s}(V)$  such that  $A(V_{\geq a}) \subset V_{\geq a+2}$  for any  $a \in \mathbf{Z}$  and such that the map  $\bar{A} \in \text{End}(\text{gr}(V_*)_2)$  induced by  $A$  belongs to  $\text{End}(\text{gr}(V_*)_2)^0$ . Using 1.4(a) we see that in our case the following statement is equivalent to A.2:

(a) *the map  $\sqcup_{V_* \in \mathfrak{F}_s(V)} \tilde{\xi}'(V_*) \rightarrow \mathcal{N}_{\mathfrak{g}}, A \mapsto A$ , is a bijection;*

If  $A \in \mathcal{N}_{\mathfrak{g}}$  then  $V_*^A \in \mathfrak{F}(V)$  (see 2.5) is self-dual and the map  $\bar{A} \in \text{End}(\text{gr}(V_*)_2)$  induced by  $A$  is in  $\text{End}(\text{gr}(V_*)_2)^0$  and is skew-adjoint with respect to  $(, )_0$  (the proofs are completely similar to those in [L2, 3.2(c)], [L2, 3.2(d)]; this implies that  $\dim \text{gr}_a(V_*)$  is even when  $a$  is even, hence  $V_*^A \in \mathfrak{F}_s(V)$ . Then  $A \mapsto (A, V_*^A)$  is a well defined map  $\mathcal{N}_{\mathfrak{g}} \rightarrow \sqcup_{V_* \in \mathfrak{F}_s(V)} \tilde{\xi}'(V_*)$  which, by [L2, 2.4] is an inverse of the map (a).

**A.4.** Let  $V, Q, (, ), G = SO(V), \mathfrak{g} = \mathfrak{o}(V)$  be as in 1.5. For any  $V_* = (V_{\geq a}) \in \mathfrak{F}_o(V)$  let  $\eta'(V_*)$  be the set of all  $A \in \mathfrak{o}(V)$  such that  $A(V_{\geq a}) \subset V_{\geq a+2}$  for any  $a \in \mathbf{Z}$  and such that the map  $\bar{A} \in \text{End}(\text{gr}(V_*)_2)$  induced by  $A$  belongs to  $\mathfrak{o}(\text{gr}(V_*)_2)^0$  (the last set is defined in terms of the grading  $\text{gr}_a(V_*)$  and the quadratic form on  $\text{gr}(V_*)$  induced by  $Q$ ). Using 1.5(a) we see that in our case the following statements are equivalent to A.2:

(a) *the map  $\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta'(V_*) \rightarrow \mathcal{N}_{\mathfrak{g}}, A \mapsto A$ , is a bijection.*

Assume first that  $p \neq 2$ . If  $A \in \mathcal{N}_{\mathfrak{g}}$  then  $V_*^A \in \mathfrak{F}(V)$  (see 2.5) is in  $\mathfrak{F}_o(V)$  (by an argument entirely similar to that in [L3, 3.3]). Then  $A \mapsto (A, V_*^A)$  is a well defined map  $\mathcal{N}_{\mathfrak{g}} \rightarrow \sqcup_{V_* \in \mathfrak{F}_o(V)} \eta'(V_*)$  which, by [L2, 2.4] is an inverse of the map (a). It remains to prove (a) in the case where  $p = 2$ .

We show that the map (a) is injective. The proof is almost the same as that in 2.9. We argue by induction on  $\dim V$ . If  $\dim V \leq 1$  the result is trivial. Assume now that  $\dim V \geq 2$ . Let  $T \in \mathfrak{o}(V)$  and let  $V_* = (V_{\geq a}), \tilde{V}_* = (\tilde{V}_{\geq a})$  be two filtrations in  $\mathfrak{F}_o(V)$  such that  $T \in \eta'(V_*)$  and  $T \in \eta'(\tilde{V}_*)$ . We must show that  $V_* = \tilde{V}_*$ . Let  $\bar{T} \in \text{End}(\text{gr}V_*)_2, \bar{T}_1 \in \text{End}(\text{gr}\tilde{V}_*)_2$  be the endomorphisms induced by  $T$ . If  $\bar{T} = 0$  or  $\bar{T}_1 = 0$  then, as in the proof in 2.9 we see that  $V_* = \tilde{V}_*$ , as desired. Thus we can assume that  $\bar{T} \neq 0, \bar{T}_1 \neq 0$ . Hence  $\text{gr}_a(V_*) \neq 0$  for some  $a \neq 0$  and  $\text{gr}_a(\tilde{V}_*) \neq 0$  for some  $a \neq 0$ . Let  $m \geq 1$  be the largest integer such that  $\text{gr}_m(V_*) \neq 0$ . Let  $\tilde{m} \geq 1$  be the largest integer such that  $\text{gr}_{\tilde{m}}(\tilde{V}_*) \neq 0$ . Using 2.8(j) we see that  $V_{\geq -m+1} = H_T = \tilde{V}_{\geq -\tilde{m}+1}$ ,  $m = \max(e_T - 1, 2f_T - 2) = \tilde{m}$ . It follows that  $m = \tilde{m}$  and  $V_{\geq -m+1} = \tilde{V}_{\geq -m+1}$ . Since  $V_*$  is a  $Q$ -filtration we have  $V_{\geq m} = \{x \in V; (x, V_{\geq -m+1}) = 0; Q(x) = 0\}$ , see [L3, 1.4(b)]. Similarly we have  $\tilde{V}_{\geq m} = \{x \in V; (x, \tilde{V}_{\geq -m+1}) = 0; Q(x) = 0\}$ . Hence  $V_{\geq m} = \tilde{V}_{\geq m}$ . Let  $V' = V_{\geq -m+1}/V_{\geq m} = \tilde{V}_{\geq -m+1}/\tilde{V}_{\geq m}$ . Note that  $V'$  has a natural nondegenerate quadratic form induced

by  $Q$ . We set  $V'_{\geq a} = \text{image of } V_{\geq a} \text{ under } V_{\geq -m+1} \rightarrow V'$  (if  $a \geq -m+1$ ),  $V'_{\geq a} = 0$  (if  $a < -m+1$ ). We set  $\tilde{V}'_{\geq a} = \text{image of } \tilde{V}_{\geq a} \text{ under } \tilde{V}_{\geq -m+1} \rightarrow V'$  (if  $a \geq -m+1$ ),  $\tilde{V}'_{\geq a} = 0$  (if  $a < -m+1$ ). Then  $V'_* = (V'_{\geq a})$ ,  $\tilde{V}'_* = (\tilde{V}'_{\geq a})$  are filtrations in  $\mathfrak{F}_o(V')$ . Also  $T$  induces an element  $T' \in \mathfrak{o}(V')$  and we have  $T' \in \eta'(V'_*)$ ,  $T' \in \eta'(\tilde{V}'_*)$ . Note also that  $\dim V' < \dim V$ . By the induction hypothesis we have  $V'_* = \tilde{V}'_*$ . It follows that  $V_{\geq a} = \tilde{V}_{\geq a}$  for any  $a \geq -m+1$ . If  $a < -m+1$  we have  $V_{\geq a} = \tilde{V}_{\geq a} = V$ . Hence  $V_* = \tilde{V}_*$ , as desired. Thus the map (a) is injective.

We show that the map (a) is surjective. The proof is somewhat similar to that in 2.10. By a standard argument we can assume that  $\mathbf{k}$  is an algebraic closure of the field  $\mathbf{F}_2$  with 2 elements. We can also assume that  $\dim V \geq 2$ . We choose an  $\mathbf{F}_2$ -rational structure on  $V$  such that  $Q$  is defined and split over  $\mathbf{F}_2$ . Then the Frobenius map relative to the  $\mathbf{F}_2$ -structure acts naturally and compatibly on the source and target of the maps (a) and 2.7(a). We denote each of these actions by  $F$ . It is enough to show that for any  $n \geq 1$  the map  $(\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta'(V_*))^{F^n} \rightarrow \mathcal{N}_{\mathfrak{g}}^{F^n}$ ,  $A \mapsto A$  is a bijection. (Here  $()^{F^n}$  is the fixed point set of  $F^n$ .) Since the last map is injective (by the previous paragraph) it is enough to show that  $|\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta'(V_*)^{F^n}| = |\mathcal{N}_{\mathfrak{g}}^{F^n}|$ . From 2.10 we see that  $|\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*)^{F^n}| = 2^{nr}$  ( $r$  as in 2.10). According to [S] we have also  $|\mathcal{N}_{\mathfrak{g}}^{F^n}| = 2^{nr}$ . It is then enough to show that

$$|\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta'(V_*)^{F^n}| = |\sqcup_{V_* \in \mathfrak{F}_o(V)} \eta(V_*)^{F^n}|$$

or equivalently that

$$\sum_{V_* \in \mathfrak{F}_o(V)^{F^n}} |\eta'(V_*)^{F^n}| = \sum_{V_* \in \mathfrak{F}_o(V)^{F^n}} |\eta(V_*)^{F^n}|.$$

It is enough to show that for any  $V_* \in \mathfrak{F}_o(V)^{F^n}$  we have  $|\eta'(V_*)^{F^n}| = |\eta(V_*)^{F^n}|$ . From the definitions we have  $|\eta(V_*)^{F^n}| = |\mathfrak{o}(\text{gr}(V_*)_2^0)|2^{nd}$  where  $d$  is the dimension of the vector space  $\{C \in \mathfrak{o}(V); C(V_{\geq a}) \subset V_{\geq a+3} \quad \forall a\}$ . From [L3, 1.5(c),(d)] we see that  $|\eta(V_*)^{F^n}| = |\mathfrak{o}(\text{gr}(V_*)_2^0)|2^{nd}$  where  $d$  is as above. We see that  $|\eta'(V_*)^{F^n}| = |\eta(V_*)^{F^n}|$ . This proves the surjectivity of the map (a) and completes the proof of A.2.

**A.5.** Assume that  $G$  is as in 2.4 and that  $\Delta, \tilde{\Delta} \in D_G$  are conjugate under  $G$ . The following statement is a strengthening of the statement that  $\Psi_{\mathfrak{g}}$  in A.2 is injective:

(a) *If  $x \in \mathfrak{s}^{\Delta}$ ,  $x \in \mathfrak{g}_{\geq 2}^{\tilde{\Delta}}$  for some  $x$  then  $\Delta = \tilde{\Delta}$ .*

The proof is essentially the same as that of 2.11(a).

**A.6.** Let  $G$  be as in A.2. Let  $\mathcal{O} \in \mathfrak{A}_G$  (see 2.3). For any  $\Delta \in D_G$  and  $\omega \in G \backslash Z_{\mathcal{O}}$  (see 2.3) let  $\mathfrak{s}_{\omega}^{\Delta}$  be the inverse image of  $\omega_{\Delta}$  (see 2.3) under the map  $\mathfrak{s}^{\Delta} \rightarrow \Sigma^{\Delta}$  in A.1; we have a partition  $\mathfrak{s}^{\Delta} = \sqcup_{\omega \in G \backslash Z_{\mathcal{O}}} \mathfrak{s}_{\omega}^{\Delta}$ . We set

$$\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}} = \Psi_{\mathfrak{g}}(\sqcup_{\Delta \in \mathcal{O}} \mathfrak{s}^{\Delta}),$$

$$\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}, \omega} = \Psi_G(\sqcup_{\Delta \in \mathcal{O}} \mathfrak{s}_{\omega}^{\Delta}), (\omega \in G \backslash Z_{\mathcal{O}}).$$

The subsets  $\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}}$  are called the *pieces* of  $\mathcal{N}_{\mathfrak{g}}$ . They form a partition of  $\mathcal{N}_{\mathfrak{g}}$  into subsets (which are unions of  $G$ -orbits) indexed by  $\mathfrak{A}_G = \mathfrak{A}_{G'}$ . The subsets  $\mathcal{N}_{\mathfrak{g}, \omega}^{\mathcal{O}}$  are called the *subpieces* of  $\mathcal{N}_{\mathfrak{g}}$ . They are in bijection with the subpieces of  $G$ .

When  $\omega$  varies in  $G \backslash Z_{\mathcal{O}}$  the subpieces  $\mathcal{N}_{\mathfrak{g},\omega}^{\mathcal{O}}$  form a partition of a piece  $\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}}$  into subsets (which are unions of  $G$ -orbits); the number of these subsets is a power of 2.

In the remainder of this subsection we assume that 2.1(c) holds. Then for each  $\mathcal{O} \in \mathfrak{A}_G$  and  $n \geq 1$ , the piece  $\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}}$  is  $F$ -stable and from the definitions we have  $|\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}}|^{F^n} = |\mathcal{O}^{F^n}| \cdot |(\mathfrak{g}_{\geq 3}^{\Delta})^{F^n}| \cdot |(\Sigma^{\Delta})^{F^n}|$  where  $\Delta$  is any point of  $\mathcal{O}^F$ . (Note that  $\mathcal{O}$  is  $F$ -stable by 2.1.) We have  $|(\mathfrak{g}_{\geq 3}^{\Delta})^{F^n}| = |G_{\geq 3}^{\Delta})^{F^n}|$ . Using this and 2.4(a) we see that  $|\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}}|^{F^n} = |\mathcal{U}_G^{\mathcal{O}}|^{F^n}$ . In particular,  $|\mathcal{N}_{\mathfrak{g}}^{\mathcal{O}}|^{F^n}$  is a polynomial in  $p^n$  with integer coefficients independent of  $p, n$ .

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